FINITE DIFFERENCE APPROXIMATION FOR STOCHASTIC PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT

Differential equations, especially partial differential equations (PDES) have wide range of applications in sciences, finance (economics), Engineering and so forth. In last decade, substantial amount of work has been done in studying stochastic partial differential equations (SPDES). A SPDE is a PDE containing a random 'noise' term. SPDES have no analytical solutions. Various numerical methods have been developed from time to time and tested for their validity using Matlab program.

In this thesis, we will discuss the finite difference method for stochastic parabolic partial differential equations. Matlab software is used for simulation of the solution of this equation. The main objective of this thesis is to investigate the finite difference approximation of a stochastic parabolic partial differential equation with white noise. We discuss alternative proof for error bounds using Green function in support of this method.

Keywords :

- **Stochastic parabolic partial differential equation.**
- **White noise.**
- **Green function.**
- **Brownian motion.**
- **Isometry property.**
- **Forward Euler method.**
- **Backward Euler method.**
- **Crank-Nicolson method.**

This work is original and has not been submitted previously in support of any qualification or course.

Signed ________________

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CONTENTS

CHAPTER 1

INTRODUCTION

In this thesis, we have discussed the finite difference method for solving the stochastic parabolic Partial differential equation.

In Chapter-4, we consider the finite difference method for deterministic parabolic PDE. The methods include forward Euler difference method, backward Euler difference method and Crank-Nicolson difference method. In this section, we have emphasized how the error estimate of time-step (k) and space-step (h) effect the results of the numerical solution. The convergence order is $O(k + h^2)$ for backward Euler method, where k = time-step size > 0, h = space-step size > 0. The simulation shows that the analytical results agree with the numerical results. A test problem is considered to justify the numerical results and analytical results which are in accordance.

In Chapter-5, we consider the finite difference method for stochastic parabolic partial differential equation. We first consider how to approximate the space-time white noise $\frac{\partial^2 W(t, x)}{\partial t\, \partial x} = \dot{W}(t, x)$. We use the piece-wise constant approach to approximate the white noise. Then we get a "new" equation which is a standard parabolic PDE. This equation can be solved using standard finite difference method like forward Euler method, backward Euler method and Crank-Nicolson method.

Further, we prove the error estimate using the Green function estimation. The convergence order is O ($k^{1/4} + h^{1/2}$) for backward Euler method, k = time-step size > 0 and h = space-step size > 0. Finally, we consider the numerical stimulation for a test problem. The numerical results and theoretical results are in accordance.

CHAPTER 2

RATIONALE

Partial differential equations (PDEs) are at the core of the mathematical modeling ([21] [24] [25] [28] [32]) of various phenomena occurring in physical Sciences, Chemistry, Biology, Engineering etc since long time. In recent years, we have found their use in other fields outside the pure mathematics [16].

During the last twenty years, there has been increasing demand for the use of stochastic calculus in fields akin advanced engineering applications, economics (finance), climate/weather predictions etc. There has been growing interest among engineers and researchers for stochastic partial differential equations (SPDEs) and their applications ([9] [12] [16] [20] [26] [29]). Very few SDEs have analytical solutions and it is also true for SPDEs. To investigate the predictions of PDE models or SPDE models of various phenomena, it is necessary to approximate their solution numerically. This has motivated scientists and mathematicians to establish and test for validity of various numerical methods in solving SPDEs ([12] [13] [15] [17]). The important numerical methods include finite difference methods and finite element methods.

Using numerical methods to generate solutions to SPDEs will help not only understanding them better way but also use in appropriate applications in relevant situations. Theory and numerical work often go together. While theory can provide useful numerical methods, numerical information (pictures) may lead in proving their validity.

Moreover, the numerical methods are also useful outside the mathematics. SPDEs are already in practical use such as in finance (Black scholes model for bonds and stocks) ([12] [16]).

Our aim is to show that how simple methods already in use for numerical solutions of PDEs (Chapter-4) can be extended to provide numerical solution of SPDEs (Chapter-5). We shall concentrate on strong solutions, that is, solution obtained for one particular realization of Brownian motion or of a Brownian sheet.

The main concept is to introduce methods that will be used to find numerical solutions to SPDEs. There exist many results of the convergence for the numerical methods for SDEs. But few results are available in case of SPDEs. Solutions in numerical experiments have properties of predictions or the subsequently obtained by analytical results. The methods we have used will be presented through concrete example the **Heat equation** with noise. We obtained the convergence order of the approximation theoretically. We also show the convergence order both in time and space numerically. The numerical results are in accordance with the theoretical results.

CHAPTER 3

SOME MATHEMATICAL PRELIMINARIES

We outline some definitions, properties of Brownian motion and Itô integrals which are essential for understanding the thesis. We recognize the range of texts available and mention the software we used. The software and the texts (see Bibliography) were fundamental to the writing of this thesis.

3.1 Texts :

While writing this thesis, a knowledge of stochastic calculus and partial differential equations is utmost important.

During the past decade there is an accelerating interest in the development of numerical methods for stochastic partial differential equations (SPDEs). We found a systematic presentation of numerical methods for the solution of stochastic partial differential equations (SPDEs) in ([10] [11] [12] [13] [14] [15]).

The applications of SPDEs in important fields such as physics, chemistry, biology, engineering, economics, ecology, hydrology, filtering, control, genetics etc. are emphasized and examples of models involving SPDEs are presented in ([10] [13] [17] [19] [22] [23]).

In general, further information and appropriate references for this thesis will be provided in the Bibliography.

3.2 Software and Programs :

The numerical computation was performed using the student edition of Matlab version 7.5.

3.3 Itô Integral :

3.3.1 Definition of Brownian Motion, B (t) :

B (t) {sometimes W (t), Wiener process} represents the position of particle at time t. A one-dimensional real-valued stochastic process B (t), $t \geq 0$ is called Brownian motion if

(1) B (0) = 0 and the function is continuous (with probability one).

(2) Stationary and for any $t_0 < t_1 < t_2 < \ldots < t_n$, the increments B_{t_0}, $B_{t_1} - B_{t_0}$, …, $B_{t_{n-1}} - B_{t_n}$ are independent.

(3) For any $s, t \geq 0$, the increments $B(t) - B(s) \sim N(0, t - s)$ is normally distributed.

We observe that

- $E[B(t)] = 0$
- $E[(B(t) - B(s))^2] = t - s$
- $[dB(t)]^2 = dt$

3.3.2 Principle of Itô Integral :

The stochastic integral $\int_0^T X(t)\, dB(t)$ also written as $\int X\, dB$ or $X \cdot B$ is called Itô Integral [16].

The Itô integral is defined as a sum $\int_0^T X(t)\, dB(t) = \lim_{n \to \infty} \sum_{i=0}^{n-1} c_i\, (B(t_{i+1}) - B(t_i))$

where $c_i = X(t_i)$

Note that :

(a) If $X(t) = 1$, then $\int_0^T dB(t) = B(T) - B(0)$

(b) If $X(t) = c$, a constant, then $\int_0^T c\, dB(t) = c\,(B(T) - B(0))$

In this way we can integrate piecewise constant processes with respect to B (t) (Brownian motion). The integral over (0, T] should be the sum of integrals over two sub-intervals (0, a] and (a, T].

Thus, if X (t) takes two values c_1 on (0, a] and c_2 on (a, T] then the integral of X (t) with respect to B (t) is easily defined.

Thus, by limiting procedure the integral is defined for more general processes. This is the principle of Itô integral [16].

3.3.3 Isometry Property : [19]

$$\mathbf{E\left[\int_0^T X(t)\, dB(t)\right]^2 = \int_0^T E\,[X^2(t)]\, dt}$$

3.3.4 Strong Solution of SDE :

Consider the stochastic differential equation

$$dX(t) = a(t, X(t))\, dt + b(t, X(t))\, dB(t) \qquad \ldots (3.1)$$

where $B(t), t \geq 0$ is a Brownian motion. $a(t, X(t))$ and $b(t, X(t))$ are deterministic functions. The solution $X(t)$, if it satisfies equation (3.1), is then a stochastic process.

Stochastic differential equation (3.1) in integral form written as :

$$X(t) = X(0) + \int_0^t a(s, X(s))\, ds + \int_0^t b(s, X(s))\, dB(s), \quad 0 \leq t \leq T \qquad \dots (3.2)$$

where the first integral on R.H.S. is a Riemann integral and the second one is an Itô stochastic integral ([16] [19]).

A strong solution $X(t), t \in (0, T]$ to the Itô stochastic differential equation satisfies the following conditions.

(1) $X(t)$ is adapted to Brownian motion, that is, at time t, $X(t)$ is $\mathcal{F}_t = \sigma(B(s), s \leq t)$ measurable.

(2) The integrals in the integral form are well defined as Riemann and Itô stochastic integrals respectively and equation (3.2) holds.

(3) $X(t)$ is a function of the underlying Brownian sample path and of the coefficient functions $a(t, X)$ and $b(t, X)$.

Thus a strong solution to equation (3.2) is based on the path of the underlying Brownian motion.

If we change the Brownian motion by another Brownian motion, we get another strong solution with the new Brownian motion in it.

3.3.5 Existence and Uniqueness Theorem for strong solutions of Itô stochastic differential equations :

Let $T > 0$ and let

$$\mu : \mathbb{R}^n \times [0, T] \to R^n;$$

$$\sigma : \mathbb{R}^n \times [0, T] \to R^{n \times m};$$

be measurable functions for which there exists constants C and D such that

$$| u (t, x) | + | \sigma (t, x) | \leq C (1 + | x |);$$

$$| u (t, x) - u (t, y) | + | \sigma (t, x) - \sigma (t, y) | \leq D | x - y |;$$

for all $t \in [0, T]$ and all x and $y \in R^n$ where

$$| \sigma^2 | = \sum_{i, j = 1}^{n} | \sigma_{ij} |^2$$

Let Z be a random variable that is independent of the σ-algebra generated by B_s, $s \geq 0$ and with finite second moment.

$$E [| Z |^2] < + \infty$$

Then the stochastic differential equation/initial value problem

$$dX_t = \mu (t, X_t) dt + \sigma (t, X_t) dB_t \text{ for } t \in [0, T];$$

$$X_0 = z$$

has a almost surely unique t-continuous strong solution $(t, \omega) \rightarrow X_t(\omega)$. The solution satisfies the following [19] :

- X_t is a function of B_s, $s \leq t$
- $E\left[\int_0^T |X_t|^2 \, dt\right] < \infty$.

CHAPTER 4

NUMERICAL SOLUTION OF PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS (DETERMINISTIC)

4.1 Introduction :

A general linear homogeneous partial differential equation of order two in two variables t, x is :

$$au_{xx} + 2bu_{xt} + cu_{tt} + du_x + eu_t + fu = 0 \quad \ldots (4.1)$$

which is parabolic in nature if $b^2 - ac = 0$.

The standard heat or diffusion equation is a parabolic partial differential equation.

u (t, x) = Temperature at any point x from left at time t.

$\frac{\partial u\,(t, x)}{\partial x}$ = Temperature gradient.

$$\frac{\partial u}{\partial t} = \sigma \frac{\partial^2 u}{\partial x^2} \quad \text{where} \quad \sigma = \frac{k}{c \cdot \mathbf{9}} \quad \ldots (4.2)$$

This is known as **canonical** form of parabolic partial differential equation, where

k = Thermal conductivity of the material of the rod (assumed constant)

c = Heat capacity (assumed constant)

9 = density of materials of the rod.

4.2 Why Numerical Methods?

The solution of a 1D heat equation $\frac{\partial u}{\partial t} = \sigma \frac{\partial^2 u}{\partial x^2}$

in the region $\Omega = \{0 \le x \le 1, t \ge 0\}$ subject to

Initial conditions : $u(0, x) = \phi(x), \quad 0 \le x \le 1$ and

Boundary conditions : $u(t, 0) = u(t, 1) = 0, \quad t \ge 0$

is a function of the form

$U_n(t, x) = \sin(n\pi x) \cdot e^{-n^2\pi^2\sigma t}$ for an integer n.

Since equation (4.2) is linear, we may use the principle of supper imposition for finite number of solutions, that is,

$$U_N(t, x) = \sum_{i=1}^{N} a_i u_i(t, x)$$

is also a solution. Where at $t = 0$, $u(0, x) = \phi(x), \quad 0 \le x \le 1$ and

we get $\phi(x) = \sum_{i=1}^{\infty} a_i \sin(i\pi x)$.

where $a_i = 2 \int_0^1 \phi(x) \sin(i\pi x)\, dx$

Provided ϕ (x) is sufficiently smooth for the convergence. Therefore, the solution is

$$u(t, x) = \sum_{i=1}^{\infty} a_i \sin(n\pi x)\, e^{-n^2\pi^2\sigma t} \qquad \ldots (4.3)$$

$$= \sum_{i=1}^{\infty} \left(2\int_0^1 \phi(y) \sin(i\pi y)\, dy \right) \cdot \sin(n\pi x) \cdot e^{-n^2\pi^2\sigma t}$$

$$= \int_0^1 \left(\underbrace{\sum_{i=1}^{\infty} 2 \sin(i\pi y) \sin(n\pi x)\, e^{-n^2\pi^2\sigma t}}_{\Downarrow \ \text{Denoting this as } G(t,x,y)} \right) \sin(i\pi y)\, dy$$

$$= \int_0^1 G(t, x, y)\, v(y)\, dy \qquad \ldots (4.4)$$

where G is called the Green's function.

4.2.1 Green's Function :

The unique solution of PDE can be written in a very compact form by introducing an auxiliary function known as the Green's function ([25] [32]).

The solution u (t, x) of the heat equation

$$u_t - u_{xx} = 0, \quad 0 \le x \le 1 \qquad \ldots (4.5)$$

$$u(0) = v(x)$$

Can be written in simple form as

$$u(t, x) = \int_0^1 G(t, x, y)\, v(y)\, dy \qquad \ldots (4.6)$$

where $$G(t, x, y) = \sum_{j=1}^{\infty} e^{-\lambda_j t} \cdot e_j(x) \cdot e_j(y)$$

where $$\lambda_j = j^2\pi^2,$$

$$e_j(x) = \sqrt{2} \sin(j\pi x)$$

The function G is called the Green's function for the equation (4.5).

4.2.2 Drawbacks :

(1) The series in equation (4.3) may converge very slowly.

(2) Hundreds of terms may be required to obtain a solution of sufficient accuracy.

(3) This technique does not generalize to other boundary conditions.

(4) This technique does not work for $\sigma = \sigma(t, x)$ or $\sigma = \sigma(t, x, y)$.

We can develop some numerical methods which not only overcomes these drawbacks but can be used to solve more difficult problems.

4.3 Explicit Methods :

In an explicit methods the new value of a quantity u can be immediately calculated from the values of u that are already known.

4.3.1 Finite Difference Methods :

Partial differential equations are solved by numerous methods. The popular methods amongst them are **finite difference methods** and **finite element methods** ([17] [29]).

We will discuss finite difference methods in this thesis.

Let

$h = \Delta x$ = equal spacing of grid work in the x-direction.

By Taylor series expansion, we get

$$f(x_n + h) = f(x_n) + hf'(x_n) + \frac{h^2}{2!} f''(x_n) + \frac{h^3}{3!} f'''(x_n) + \frac{h^4}{4!} f^{(4)}(\xi_1) \qquad \dots (4.7)$$

where $x_n < \xi_1 < x_n + h$

Similarly,

$$f(x_n - h) = f(x_n) - hf'(x_n) + \frac{h^2}{2!} f''(x_n) - \frac{h^3}{3!} f'''(x_n) + \frac{h^4}{4!} f^{(4)}(\xi_2) \qquad \dots (4.8)$$

where $x_n - h < \xi_2 < x_n$

Now using equations (4.7) and (4.8), we approximate first derivative as :

$$\frac{f(x_n + h) - f(x_n - h)}{2h} = f'(x_n) + \frac{h^3}{3!} f'''(\xi) \qquad \dots (4.9)$$

where $x_n + h < \xi < x_n - h$

Let $f_{n+1} = f(x_n + h)$, $f_n = f(x_n)$ and $f_{n-1} = f(x_n - h)$. Then, equation (4.9) can be written as

$$\frac{f_{n+1} - f_{n-1}}{2h} = f'_n + O(h^2) \qquad \dots (4.10)$$

An approximation relation in equation (4.10) can be written as

$$f'_n \approx \frac{f_{n+1} - f_{n-1}}{2h} \qquad \text{… (4.11)}$$

Similarly, the second derivative can be approximated by

$$\frac{f(x_n + h) - 2f(x_n) + f(x_n - h)}{h^2} = f''(x_n) + O(h^2) \qquad \text{… (4.12)}$$

that is

$$f''_n \approx \frac{f_{n+1} - 2f_n + f_{n-1}}{h^2} \qquad \text{… (4.13)}$$

Now, we shall approximate the partial derivatives for the function $u = u(x, y)$ at (x_i, y_i) for $i = 1, 2, \ldots$ where $u_{i,j} = u(x_i, y_j)$.

$$\frac{\partial^2 u}{\partial x^2}(x_i, y_j) \approx \frac{u(x_{i+1}, y_j) - 2u(x_i, y_j) + u(x_{i-1}, y_j)}{(\Delta x)^2} \qquad \text{… (4.14)}$$

OR

$$\frac{\partial^2 u}{\partial x^2}(x_i, y_j) \approx \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} \qquad \text{… (4.15)}$$

Similarly first order and second order partial derivatives are as follows :

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{(x_i, y_j)} \approx \frac{u_{i,j+1} - 2u_{i,j} + 2u_{i,j-1}}{k^2} \qquad \text{… (4.16)}$$

$$\left(\frac{\partial u}{\partial x}\right)_{(x_i, y_j)} \approx \frac{u_{i+1,j} - u_{i-1,j}}{h^2} \qquad \text{… (4.17)}$$

$$\left(\frac{\partial u}{\partial y}\right)_{(x_i,\, y_j)} \approx \frac{u_{i,\,j+1} - u_{i,\,j-1}}{k^2} \qquad \ldots (4.18)$$

where $h = \Delta x$ and $k = \Delta y$

Next, we substitute these partial derivatives by the difference relations equations (4.15) through (4.18) into the given partial differential equation. The system in partial differential equation is then transformed into difference equations. The solution of these difference equations is the solution of the given partial differential equation.

By substituting the finite difference relations into the 1D space heat equation (4.2), we get

$$\frac{u_{i,\,j+1} - u_{i,\,j}}{\Delta t} = \sigma\, \frac{u_{i+1,\,j} - 2u_{i,\,j} + u_{i-1,\,j}}{(\Delta x)^2}$$

OR

$$u_{i,\,j+1} = \sigma\, \frac{\Delta t}{(\Delta x)^2}\,(u_{i+1,\,j} + u_{i-1,\,j}) + \left(1 - 2\sigma\frac{\Delta t}{(\Delta x)^2}\right) u_{i,\,j}$$

where $\left.\frac{\partial u}{\partial t}\right|_{\substack{x = x_i \\ t = t_j}} = \frac{u_{i,\,j+1} - u_{i,\,j}}{\Delta t}$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{\substack{x = x_i \\ t = t_j}} = \frac{u_{i+1,\,j} - 2u_{i,\,j} + u_{i-1,\,j}}{(\Delta x)^2}$$

Let $r = \sigma \cdot \frac{\Delta t}{(\Delta x)^2}$, then we can rewrite as :

$$u_{i,\,j+1} = r\,(u_{i+1,\,j} + u_{i-1,\,j}) + (1 - 2r)\, u_{i,\,j} \qquad \ldots (4.19)$$

Subject to

Initial condition : $u(0, x) = \phi(x) \quad a \le x \le b$ and

Boundary conditions : $u(t, a) = c_1$,

$u(t, b) = c_2, \quad t > 0.$

Note that the formula in equation (4.19) works when $0 < r \le \frac{1}{2}$.

Suppose there are $(N + 1)$ points between a and b then

$$x_i = a + \frac{b - a}{N} \cdot i$$

and $u_{i, o} = \phi(x_i)$ for $i = 0, 1, 2, \ldots, N$

$u_{o, j} = c_1$ and $U_{N, j} = c_2$ for $j = 1, 2, \ldots$

4.3.2 Forward Difference Methods :

Consider 1D heat equation with homogeneous Dirichlet boundary conditions ([2] [5] [9] [17]).

$$u_t - u_{xx} = 0, \quad x \in (0, 1), \quad t > 0 \qquad \ldots (4.20)$$

Boundary conditions : $u(t, 0) = u(t, 1) = 0$ and

Initial conditions : $u(0, x) = u_0(x)$

One method to numerically solve this problem is to approximate all the derivatives by finite differences. We partition the domain in space using a mesh $x_0, \ldots, x_j$ and in time using a mesh $t_0, \ldots, t_N$. We assume a uniform partition both in space and in time so the difference between two consecutive space points will be h and that between two consecutive time points will be k. The points

$$u_j^n \approx u\,(t_n, x_j)$$

will represent the numerical approximation of $u\,(t_n, x_j)$.

Now, using a forward difference at time t_n and a second-order central difference for the space derivative at position x_j ("FTCS"), we get the recurrence equation

$$\frac{u_j^{n+1} - u_j^n}{k} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2}$$

This is an explicit method for solving the one-dimensional heat equation.

We can find u_j^{n+1} from the other values as below :

$$u_j^{n+1} = (1 - 2r)\,u_j^n + r\left(u_{j-1}^n + u_{j+1}^n\right) \qquad \ldots (4.21)$$

where $$r = \frac{k}{h^2}$$

By knowing the values at time t_n, we can find the corresponding values at time t_{n+1} using the above recurrence relation (4.21). u_o^n and u_j^n are boundary conditions and in this case both are equal to zero.

This explicit method is known to be numerically **stable** and **convergent** whenever $r \le \frac{1}{2}$.

The numerical errors satisfy the following :

$$\left| u_j^n - u\,(t_n, x_j) \right| = O\,(k) + O\,(h^2) \quad \text{for} \quad t_n = 1 \ \text{ and fixed j.}$$

O (k) means the convergence order is one with respect to the step k and O (h^2) means the convergence order is two with respect to the step h.

The stencil for explicit forward finite difference method is as shown in Fig. 4.1.

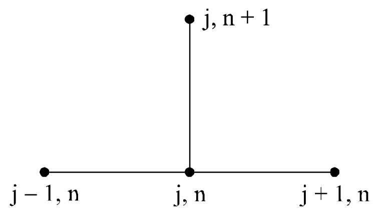

Fig. 4.1 The forward difference stencil for the finite difference method

4.3.3 Algorithm :

Explicit scheme : Finite difference method

Solve a parabolic partial differential equation

$$\frac{\partial u}{\partial t} = \sigma \cdot \frac{\partial^2 u}{\partial x^2} \ \text{over } a \le x \le b, \quad 0 \le t \le T.$$

With **Initial condition :** $u\,(0, x) = f\,(x)$ for $a \le x \le b$ and

Boundary conditions : $u\,(t, a) = c_1$ and

$u\,(t, b) = c_2$ where c_1 and c_2 are constants.

Input :

$\frac{\partial u}{\partial t} = \sigma \cdot \frac{\partial^2 u}{\partial x^2}$ and a, b, T, σ, c_1, c_2, N, M where M and N are the number of discrete points from x and t respectively.

Output :

$u(t, x)$ for $a \le x \le b,\ 0 \le t \le T$

Step 1 : [Input parameters, initial and boundary conditions]

Read $a, b, T, \sigma, c_1, c_2, N, M$

Step 2 : [Compute for initialization]

$$h = \frac{b-a}{N},\quad k = \frac{T}{M},\quad r = \sigma \cdot \frac{k}{h^2},\quad s = 1 - 2r$$

Step 3 : [Initialize boundary points]

for $i = 0$ to N do

$x_i = a + ih,\quad u_{i,0} = f(x_i)$

end do

for $j = 1$ to M do

$u_{o,j} = c_1,\quad U_{N,j} = c_2$

end do

Step 4 : [Compute other points from difference equation]

for $j = 0$ to M do

for $i = 1$ to N do

Compute $U_{i,j+1} = r\,(u_{i+1,j} + u_{i-1,j}) + (1 - 2r)\,u_{i,j}$

Step 5 : [Output]

Print $U_{i,j}$ for $i = 1$ to N, $j = 1$ to M

Step 6 : [Terminate the algorithm]

Stop.

4.4 Implicit Method :

In an implicit method, it is necessary to compute an unknown pivotal value for the solution of a system of linear or non-linear partial differential equations ([17] [19] [24]).

4.4.1 Advantages of Implicit Methods Over Explicit Methods :

In the explicit method, the time-step (Δt) has to be necessarily very small since the explicit method is only valid for $0 \le r \le \frac{1}{2}$, that is,

$$r = \frac{\Delta t}{(\Delta x)^2} \cdot \sigma$$

and attains reasonable accuracy for small Δx. On the other hand, the implicit method of an iterative method is stable and also converges for all finite values of r.

4.4.2 Crank-Nicolson Method :

4.4.2.1 Simple Version :

If we use the backward difference at time and a second-order central difference for the space derivative (‘BTCS’) in the heat equation (4.2), we get

$$\frac{u_{i,\,j} - u_{i,\,j-1}}{\Delta t} = \sigma \cdot \frac{u_{i+1,\,j} - 2u_{i,\,j} + u_{i-1,\,j}}{(\Delta x)^2}$$

OR

$$(u_{i,\,j} - u_{i,\,j-1})\,\frac{(\Delta x)^2}{\sigma \cdot \Delta t} = u_{i+1,\,j} - 2u_{i,\,j} + u_{i-1,\,j}$$

OR

$$su_{i,\,j-1} = -u_{i+1,\,j} + ru_{i,\,j} - u_{i-1,\,j} \qquad \ldots (4.22)$$

where $s = \dfrac{(\Delta x)^2}{\sigma \cdot \Delta t}$ and $r = 2 + s$ for $i = 2, 3, \ldots, n - 1$.

(a) Implicit Stencil :

An implicit stencil for the simple version of Crank-Nicolson method is shown in Fig. 4.2.

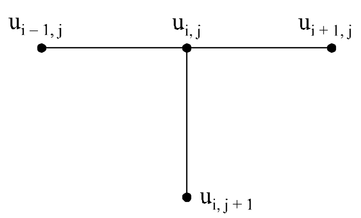

Fig. 4.2 Implicit stencil (simple version of Crank-Nicolson method)

Initial condition : $u_{i,\,1} = f(x_i)$ for $i = 2, 3, \ldots, n - 1.$

Boundary conditions : $u_{1,\,j-1} = u_{1,\,j} = c_1$ and

$$u_{n,\,j-1} = u_{n,\,j} = c_2$$

(b) Matrix Equation :

Now the matrix form of equation (4.22) is $AX = B$ where

$$A = \begin{pmatrix} r & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & r & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & r & -1 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & -1 & r & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & r \end{pmatrix}, \quad X = \begin{bmatrix} u_{2, j} \\ u_{3, j} \\ u_{4, j} \\ \vdots \\ u_{n-2, j} \\ u_{n-1, j} \end{bmatrix}$$

$$B = \begin{bmatrix} c_1 + su_{2, j-1} \\ su_{3, j-1} \\ su_{4, j-1} \\ \vdots \\ su_{n-2, j-1} \\ c_2 + su_{n-1, j-1} \end{bmatrix}$$

This system of equations can be solved by using the method of system of linear equations.

4.4.2.2 Alternative Version :

The partial derivative $\dfrac{\partial^2 u}{\partial x^2}$ of heat equation in equation (4.2) is replaced by the arithmetic mean of its central-difference approximations on the (j + 1)th and jth rows (along time scale) and the difference quotient

$\dfrac{(u_{i, j+1} - u_{i, j})}{\Delta t}$ is a central difference approximation to $\dfrac{\partial u}{\partial t}$. Then

$$\frac{\partial u}{\partial t} = \sigma \cdot \frac{\partial^2 u}{\partial x^2}$$

is approximated by

$$\frac{u_{i, j+1} - u_{i, j}}{\Delta t} = \frac{1}{2}\,\sigma \left[\frac{u_{i+1, j} - 2u_{i, j} + u_{i-1, j}}{(\Delta x)^2} + \frac{u_{i+1, j+1} - 2u_{i, j+1} + u_{i-1, j+1}}{(\Delta x)^2} \right] \quad \ldots (4.23)$$

On rearrangement, we get the Crank-Nicolson formula as :

$$-ru_{i-1,\,j+1} + (2 + 2r)\,u_{i,\,j+1} - ru_{i+1,\,j+1} = ru_{i-1,\,j} + (2 - 2r)\,u_{i,\,j} + ru_{i+1,\,j} \qquad \ldots (4.24)$$

OR

$$-u_{i-1,\,j+1} + \left(\frac{2}{r} + 2\right) u_{i,\,j+1} - u_{i+1,\,j+1} = u_{i-1,\,j} + \left(\frac{2}{r} - 2\right) u_{i,\,j} + u_{i+1,\,j}$$

where

$r = \sigma \cdot \dfrac{\Delta t}{(\Delta x)^2}$ and it may be any real finite value for $i = 2, 3, \ldots, n - 1$ and

$j = 1, 2, \ldots, m$ (say)

Here

Initial condition : $u_{i,\,1} = f(x_i)$ for $i = 2, 3, \ldots, n - 1$ and

Boundary conditions : $u_{1,\,j-1} = u_{1,\,j} = c_1$

$u_{n,\,j-1} = u_{n,\,j} = c_2$ for $j = 1, 2, \ldots, m$.

The initial given conditions of the solution map for u is given below :

i ↓ / j →	**1**	**2**	**3**	**4**	**5**	**6**	**7**
1	c_1	c_1	c_1	c_1	c_1	c_1	c_1
2	$f(x_2)$	$u_{2,\,2}$	$u_{2,\,3}$				
3	$f(x_3)$	$u_{3,\,2}$	$u_{3,\,3}$				
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
n – 1	$f(x_{n-1})$	$u_{n-1,\,2}$	$u_{n-1,\,3}$				
n	c_2	c_2	c_2	c_2	c_2	c_2	c_2

For $j = 1$, all the right hand side components of equation (4.24) are known. Therefore we obtain a system of linear equations for n unknowns $u_{i,2}$ for $i = 1, 2, \ldots, n$ as follows :

Assume $U_{0,j} = 0$ $U_{n+1,j} = 0$ for $j = 1, 2, \ldots$

when $j = 1, \ i = 1$:

$$-u_{0,2} + \left(\frac{2}{r} + 2\right) u_{1,2} - u_{2,2} = u_{0,1} + \left(\frac{2}{r} - 2\right) u_{1,1} + u_{2,1}$$

whenj = 1, i = 2 :

$$-u_{1,2} + \left(\frac{2}{r} + 2\right) u_{2,2} - u_{3,2} = u_{1,1} + \left(\frac{2}{r} - 2\right) u_{2,1} + u_{3,1}$$

when $j = 1, \ i = 3$:

$$-u_{2,2} + \left(\frac{2}{r} + 2\right) u_{3,2} - u_{4,2} = u_{2,1} + \left(\frac{2}{r} - 2\right) u_{3,1} + u_{4,1}$$

$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$

when $j = 1, \ i = n - 1$:

$$-u_{n-2,2} + \left(\frac{2}{r} + 2\right) u_{n-1,2} - u_{n,2} = u_{n-2,1} + \left(\frac{2}{r} - 2\right) u_{n-1,1} + u_{n,1}$$

when $j = 1, \ i = n$:

$$-u_{n-1,2} + \left(\frac{2}{r} + 2\right) u_{n,2} - u_{n+1,2} = u_{n-1,1} + \left(\frac{2}{r} - 2\right) u_{n,1} + u_{n+1,1}$$

Therefore we obtain a tri-diagonal system of equations as $AX = B$, where

$$A = \begin{bmatrix} \left(\frac{2}{r}+2\right) & -1 & \cdots & \cdots & \cdots & \cdots \\ -1 & \left(\frac{2}{r}+2\right) & -1 & \cdots & \cdots & \cdots \\ \cdots & -1 & \left(\frac{2}{r}+2\right) & -1 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & -1 & \left(\frac{2}{r}+2\right) & -1 \\ \cdots & \cdots & \cdots & \cdots & -1 & \left(\frac{2}{r}+2\right) \end{bmatrix}_{n \times n}$$

$$X = \begin{bmatrix} u_{1,2} \\ u_{2,2} \\ u_{3,2} \\ \vdots \\ u_{n-1,2} \\ u_{n,2} \end{bmatrix}_{n \times 1} \quad \text{and} \quad B = \begin{bmatrix} u_{0,1} + \left(\frac{2}{r}-2\right) u_{1,1} + u_{2,1} \\ u_{1,1} + \left(\frac{2}{r}-2\right) u_{2,1} + u_{3,1} \\ u_{2,1} + \left(\frac{2}{r}-2\right) u_{3,1} + u_{4,1} \\ \vdots \\ u_{n-2,1} + \left(\frac{2}{r}-2\right) u_{n-1,1} + u_{n,1} \\ u_{n-1,1} + \left(\frac{2}{r}-2\right) u_{n,1} + u_{n+1,1} \end{bmatrix}_{n \times 1}$$

$$X^T = (u_{1,2} \quad u_{2,2} \quad u_{3,2} \ldots u_{n,2}), \quad u_{1,2} = c_1, \quad u_{n,2} = c_2$$

Note :

The left-hand side of equation (4.24) contains three unknown pivotal values of u whereas the right-hand side values are known. Therefore, this is termed as an implicit method in contrast to the explicit method.

If we choose $r = 1$ then $\Delta t = \dfrac{(\Delta x)^2}{\sigma}$ and

$$-u_{i-1,j+1} + 4u_{i,j+1} - u_{i+1,j+1} = u_{i-1,j} + u_{i+1,j} \qquad \ldots (4.25)$$

for $i = 2, 3, \ldots, n-1$ and $j = 1, 2, 3, \ldots, m$ (say)

(a) Alternative Stencil :

An alternative stencil for the simple version of Crank-Nicolson method is shown in Fig. 4.3.

Initial condition : $u_{i,1} = f(x_i)$ for $i = 2, 3, \ldots, n-1$

Boundary conditions : $u_{1,j} = u_{1,j+1} = c_1$ and

$u_{n,j} = u_{n,j+1} = c_2$ for $j = 1, 2, \ldots$

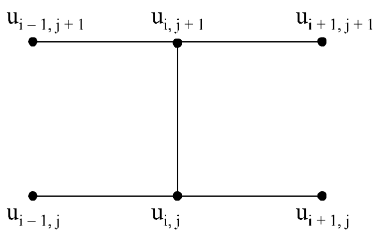

Fig. 4.3 A stencil for alternative version of Crank-Nicolson method

(b) Matrix Equation :

The matrix form of equation (4.25) is $AX = B$ where

$$A = \begin{bmatrix} 4 & -1 & 0 & \cdots & \cdots & \cdots & 0 \\ -1 & 4 & -1 & \cdots & \cdots & \cdots & 0 \\ 0 & -1 & 4 & -1 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & -1 & 4 & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & -1 & 4 \end{bmatrix}_{n \times n}$$

$$X = \begin{bmatrix} u_{2,j+1} \\ u_{3,j+1} \\ u_{4,j+1} \\ \vdots \\ u_{n-2,j+1} \\ u_{n-1,j+1} \end{bmatrix}_{n \times 1} \quad \text{and} \quad B = \begin{bmatrix} 2c_1 + u_{3,j} \\ u_{2,j} + u_{4,j} \\ u_{3,j} + u_{5,j} \\ \cdots \;\; \cdots \\ \cdots \;\; \cdots \\ u_{n-3,j} + u_{n-3,j} \\ u_{n-2,j} + 2c_2 \end{bmatrix}_{n \times 1}$$

This system of equations can be solved by using the method of system of linear equations.

Note : The Crank-Nicolson method is unconditionally stable and has better truncation error than the basic implicit method.

4.4.3 Error Estimates :

Local errors :

It is the difference between the result obtained from the method and the exact solution, provided there is no error in earlier steps.

Truncation errors :

They are obtained when an iterative method is terminated and the approximate solution differs from the exact solution.

Round-off errors :

They occur due to lack of precision in rounding of decimal quantities by computer.

Discretization errors :

They occurred when the solution of the discrete problem does not co-incide with the solution of the continuous problem.

Global errors :

The above mentioned first three errors together referred to as global errors.

The study of the generation and propagation of errors is the main aspect of numerical methods. When approximating a given problem, it is essential to understand the behavior of the error in the computed solution, its size and eventually its analytical form in order to minimize and estimate as well.

Consistency :

The finite difference method is consistent if the limit of the local truncation error is zero as h and/or k approach zero.

In all forward, backward Euler methods and C-N method, we have

$$\lim_{h \to 0} \tau_h = 0 \quad \text{where } \tau_h = \text{local truncation error.}$$

Therefore, they are all consistent.

The condition $k \to 0$ and $h \to 0$ is necessary for ensuring that the global discretization should also tends to zero. This condition is called consistency.

The method is consistent and of order p if $\| \tau_h \| = 0\ (h^p)$ for some integer $p > 0$. Consistency is only a necessary condition for convergence.

Stability :

The concept of stability refers to the sensitivity of the solution to small changes in the data or the given parameters of the problem. It is clear from numerical test problems that the finite difference method depends on k even if it is consistent.

An algorithm is numerically stable if an error, once it is generated, does not grow too much during the computation. This is only possible if the problem is well-defined. An algorithm is numerically stable if it solves a nearby problem approximately.

Convergence :

A numerical method is said to be convergent if the numerical solution approaches the exact solution as the step-size goes to zero. In other words, the global error approaches to zero as k and h approaches to zero.

Mathematically,

$$\lim_{h \to 0^+} \max_{n = 0, 1, \ldots, N} |u_j^n - u(t_n, x_j)| = 0$$

A consistent and stable finite difference method is convergent.

OR

Consistency + Stability = Convergence

We consider the one-dimensional (normalized) heat equation with homogeneous Dirichlet boundary conditions.

$$u_t = u_{xx}$$

$$u(t, 0) = u(t, 1) = 0$$

$$u(0, x) = u_0(x)$$

We discretize the space domain using $x_0, x_1, \ldots, x_j$ and time-domain using $t_0, t_1, \ldots, t_n$. h = space-space size > 0, k = time-step size > 0. We assume uniform mesh. The points $U_j^n \sim U(t_n, x_j)$ represent the numerical solution.

(a) Forward Euler scheme :

Under some stability condition, that is, $\frac{k}{h^2} < \frac{1}{2}$

$$\| u(t_n) - u^n \|_{L_2} = \left[\frac{1}{M} \sum_{j=1}^{M} |u(t_n, x_j) - u_j^n|^2 \right]^{1/2} \leq ch^2$$

where $u^n = \begin{bmatrix} u_1^n \\ u_2^n \\ \vdots \\ u_M^n \end{bmatrix}$ and $u(t_n) = \begin{bmatrix} u(t_n, x_1) \\ u(t_n, x_2) \\ \vdots \\ u(t_n, x_j) \end{bmatrix}$

(b) Backward Euler scheme :

$$\| u(t_n) - u^n \|_{L_2} = \left[\frac{1}{M} \sum_{j=1}^{M} |u(t_n, x_j) - u_j^n|^2 \right]^{1/2} \leq c\,(k + h^2)$$

The algorithm is unconditionally stable and convergent but usually more numerically intensive than the explicit method as it requires solving a system of linear equations on each time-step.

(c) Crank-Nicolson scheme :

$$\| u(t_n) - u^n \|_{L_2} = \left[\frac{1}{M} \sum_{j=1}^{M} |u(t_n, x_j) - u_j^n|^2 \right]^{1/2} \leq c\,(k^2 + h^2)$$

The Crank-Nicolson method is more accurate for small time-steps as the errors are $O(k^2 + h^2)$. However near the boundaries, the error is often only $O(h^2)$. The implicit method works the best for larger time-steps.

4.4.4 Algorithm :

Implicit scheme : Crank-Nicolson method

Solve a parabolic partial differential equation

$$\frac{\partial u}{\partial t} = \sigma \cdot \frac{\partial^2 u}{\partial x^2} \quad \text{over} \quad a \leq x \leq b, \quad 0 \leq t \leq T$$

With **Initial condition :** $u(0, x) = f(x)$ for $a \le x \le b$ and

Boundary conditions : $u(t, a) = c_1$ and

$$u(t, b) = c_2$$

Input : $\dfrac{\partial u}{\partial t} = \sigma \cdot \dfrac{\partial^2 u}{\partial x^2}$ and $a, b, T, \sigma, c_1, c_2, N, M$ where

N and M are the number of points in the interval of x and t respectively.

Output : $u(t, x)$ for $a \le x \le b$, $0 \le t \le T$

Step 1 : [Input parameters, initial and boundary conditions]

Read $a, b, T, \sigma, c_1, c_2, N, M$.

Step 2 : [Compute for initialization]

$$h = \frac{b-a}{N}, \quad k = \frac{T}{M}, \quad r = \sigma \cdot \frac{k}{h^2}$$

Step 3 : [Initialize boundary points]

for $i = 2$ to $N - 1$ do

$x_i = a + ih$; $u_{i,0} = f(x_i)$

end do

for $j = 1$ to M do

$u_{0,j} = c_1$, $u_{N,j} = c_2$

end do

Step 4 : [Prepare and solve tri-diagonal system for $U_{i,j}$ i.e. AX = B]

for j = 2 to M do

begin

Step 4.1 : [Prepare B vector]

for i = 1 to N do

compute $Bi = u_{i-1,1} + \left(\frac{2}{r} - 2\right) u_{i,1} + u_{i+1,1}$

Step 4.2 : [Prepare A matrix]

for i = 1 to N do

compute diagonal $a_{i,i} = 2 + \frac{2}{r}$

for i = 2 to N do

compute left diagonal $a_{i-1,i} = -1$

for i = 1 to N – 1 do

compute left diagonal $a_{i,i+1} = -1$

Step 4.3 : Solve tri-diagonal system of linear equations AX = B

end

Step 5 : [Output $U_{i,j}$]

for i = 1 to N

print $U_{i,j}$ for j = 1 to M

Step 6 : [Terminate the algorithm]

stop.

4.4.5 Numerical Investigations (Deterministic) :

- Test Problem :

In this section, we will consider the numerical simulation for the following problem.

$$u_t - u_{xx} + bu = f + g \quad \ldots(4.26)$$

$$u(0, x) = 10\, x^2 (1 - x)^2$$

$$u(t, 0) = u(t, 1) = 0, \quad t \geq 0$$

we will consider the deterministic case when $f = 0$

$$g = 10(1 + b)\, x^2 (1 - x)^2 e^t - 10(2 - 12x + 12x^2)\, e^t$$

$$b = 0.5$$

The exact solution of equation (4.26) is

$$u(t, x) = 10\, x^2 (1 - x)^2 e^t \quad \ldots (4.27)$$

Now we will consider the back-ward Euler method

$$\frac{U_j^n - U_j^{n-1}}{k} = \frac{U_{j-1}^n - 2U_j^n + U_{j+1}^n}{h^2} + bU_j^n = g_j^n$$

where $j = 1, 2, \ldots, M$

By theorem we have, for $t_n = 1$

$$\|e^n\| = \left[\frac{1}{M} \sum_{j=1}^{M} |u_j^n - u(t_n, x_j)|^2 \right]^{1/2} \leq c\,(k + h^2) \quad \ldots (4.28)$$

Next, we will check the convergence order for k and h respectively as follow :

Case 1 : Check for the convergence order for k :

Choose $h = 2^{-10}$

Choose $k_j = \dfrac{1}{2^j}$, $j = 1, 2, \ldots, 8$

We get $\|e^n(k_1)\|, \|e^n(k_2)\|, \|e^n(k_3)\|, \ldots, \|e^n(k_8)\|$

Since h is very small, the error is dominated by k and we can neglect h.

The ratio of norms of the errors, that is,

$$\frac{\|e^n(k_j)\|}{\|e^n(k_{j+1})\|} \approx \frac{k_j}{k_{j+1}} = \frac{\frac{1}{2^j}}{\frac{1}{2^{j+1}}} \approx 2$$

The results for $j = 1, 2, \ldots, 8$ are summarized as in below Table-1.

Table-1

j	**k_j**	**h**	**$\|e^n(k_j)\|$**	**$\dfrac{\|e^n(k_j)\|}{\|e^n(k_{j+1})\|}$**
1.0000	0.5000	0.0010	0.0203	1.3393
2.0000	0.2500	0.0010	0.0151	2.3004
3.0000	0.1250	0.0010	0.0066	2.1041
4.0000	0.0625	0.0010	0.0031	2.0190
5.0000	0.0313	0.0010	0.0015	1.9792
6.0000	0.0156	0.0010	0.0008	1.9413
7.0000	0.0078	0.0010	0.0004	1.8832
8.0000	0.0039	0.0010	0.0002	1.8832

Since $\|e^n\| \leq c\,(k + h^2) \approx ck$, neglecting h.

Taking logarithm of both sides of this inequality, we get

$$\log \|e^n\| \approx \log c + 1 \cdot \log (k)$$

The graph of $\log \|e^n\|$ against $\log (k)$ is a straight line having slope equal to one as shown in Fig. 4.4.

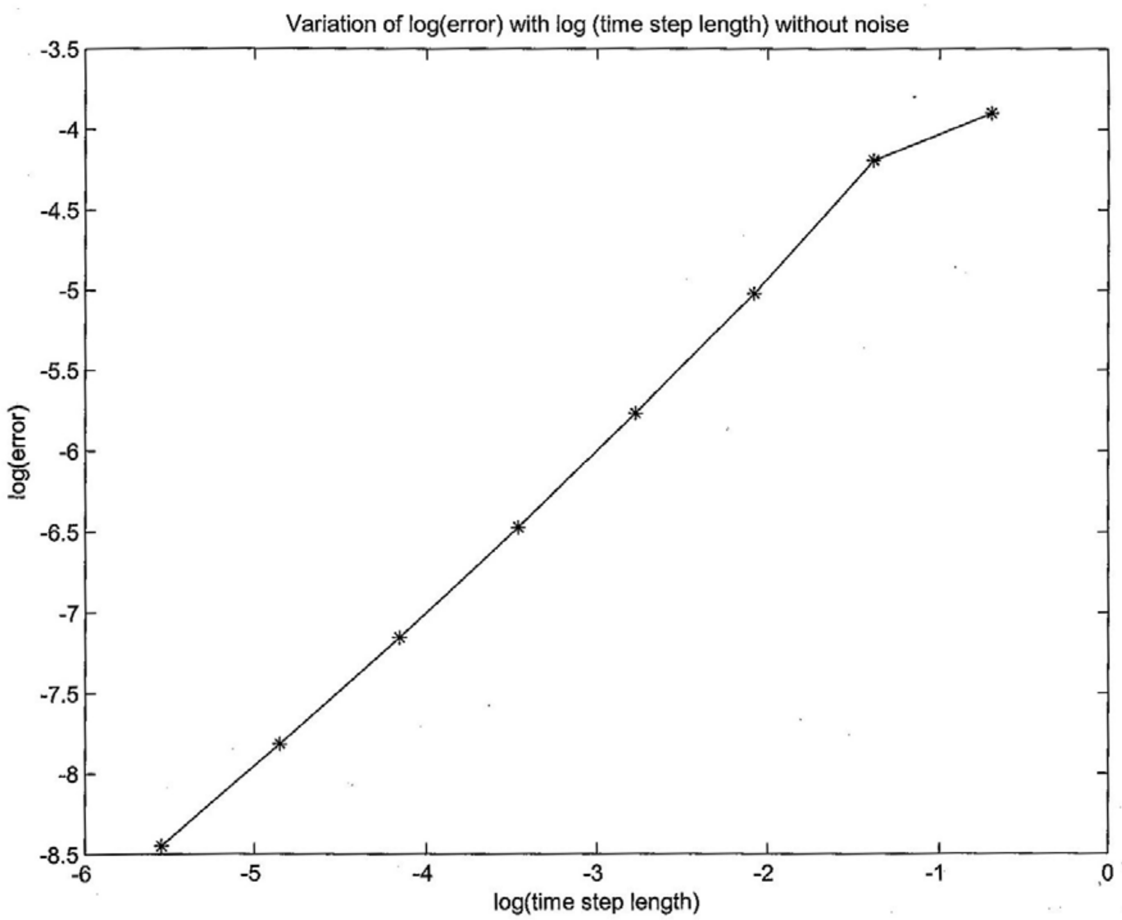

Fig. 4.4

Case 2 : Check for the convergence order for h

Choose $k = 2^{-16}$

Choose $h_j = \frac{1}{2^j}$, $j = 1, 2, 3, \ldots, 8.$

The error is dominated by h as k is very small and we get

$\log \|e^n (h_1)\|,\ \log \|e^n (h_2)\|, \ldots,\ \log \|e^n (h_8)\|$

The ratio of norms of the errors is

$$\frac{\|e^n (h_j)\|}{\|e^n (h_{j+1})\|} \approx \frac{h_j^2}{h_{j+1}^2} \approx \frac{\left(\frac{1}{2_j}\right)^2}{\left(\frac{1}{2_{j+1}}\right)^2} \approx 4$$

The results are summarized in below Table-2.

Table-2

J	**k**	**h_j**	**$\|e^n (h_j)\|$**	**$\frac{\|e^n (h_j)\|}{\|e^n (h_{j+1})\|}$**
2.0000	2^{-16}	0.2500	0.2669	3.9713
3.0000	2^{-16}	0.1250	0.0672	3.9943
4.0000	2^{-16}	0.0625	0.0168	3.9982
5.0000	2^{-16}	0.0313	0.0042	3.9976
6.0000	2^{-16}	0.0156	0.0011	3.9918
7.0000	2^{-16}	0.0078	0.0003	3.9676
8.0000	2^{-16}	0.0039	0.0001	3.9676

Since $\|e^n\| \le c\,(k + h^2) \approx ch^2$, neglecting k.

Taking logarithm of both sides of this inequality, we get

$$\log \|e^n\| \approx \log c + 2 \log (h)$$

The graph of $\log \|e^n\|$ against $\log (h)$ is a straight line having slope nearly equal to 2 as shown in Fig. 4.5.

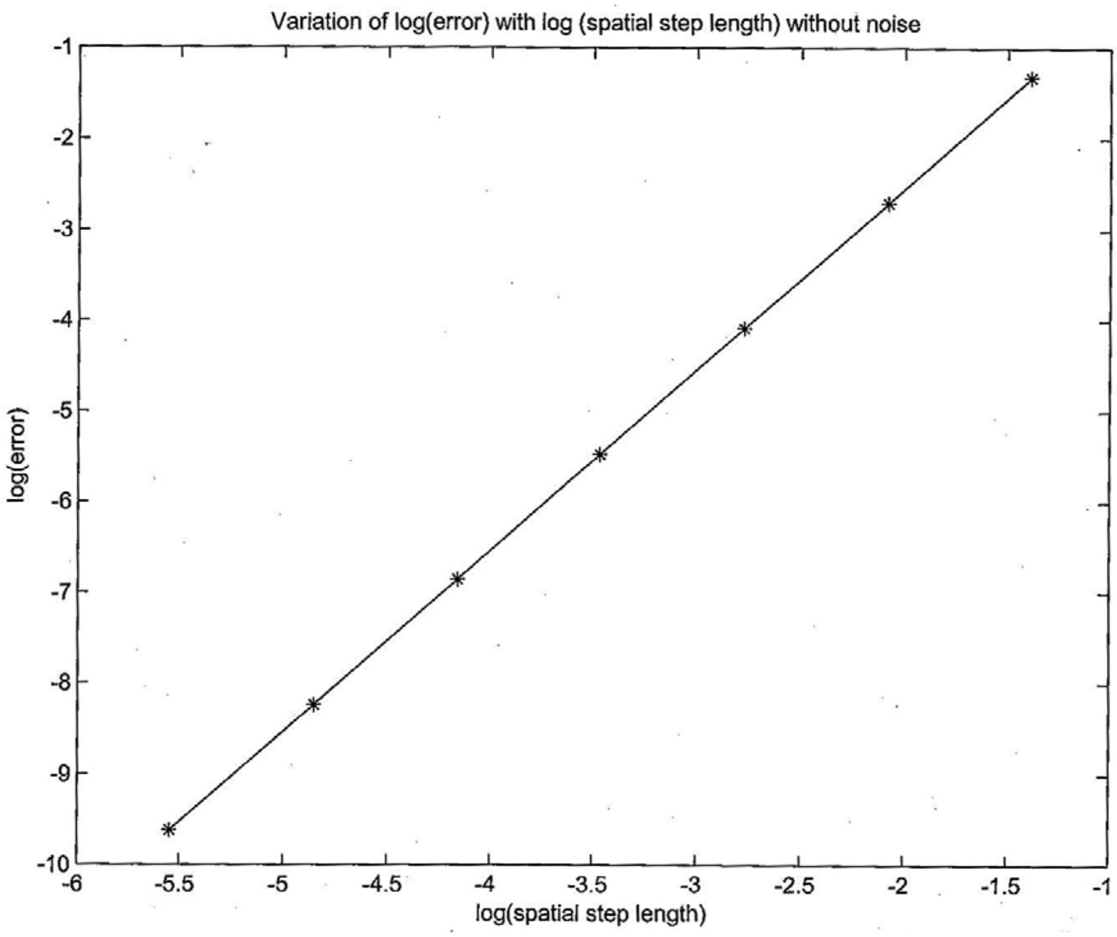

Fig. 4.5

CHAPTER 5

NUMERICAL SOLUTION OF STOCHASTIC PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

5.1 Introduction :

Stochastic partial differential equations (SPDEs) are resulted from the perturbation of deterministic equations by stochastic terms (such as W (t) or B (t), wiener process or Brownian process).

Consider an infinitesimally thin perfectly even wire of length 'L' lying flat to fulfill (0, L] condition. Let $F(t, x)$ be the amount of pressure per unit length applied in the direction of the y-axis at a point $x \in [0, L]$ to vibrate it. [In this case $F > 0$ means force is applied up along $y = +\infty$ direction and $F < 0$ means exactly in opposite direction]. Then, the position $u(t, x)$ of the wire is governed by the PDE.

$$\frac{\partial^2 u(t, x)}{\partial t^2} = k \cdot \frac{\partial^2 u(t, x)}{\partial x^2} + F(t, x) \qquad \dots (5.1)$$

Where $t \geq 0$, $0 \leq x \leq L$ and k is a physical constant. It depends on the linear mass density and the tension of the wire. Equation (5.1) is called one-dimensional wave equation. Its solution is obtained by the method of separation of variables and principle of superposition.

If ‘F’ is “random noise” or “white noise” or “stochastic term”, the equation (5.1) is no longer deterministic. It is now interpreted as an infinite dimensional integral equation ([9] [26] [29]). This has motivated scientists and mathematicians for in-depth study of stochastic partial differential equations.

Stochastic initial-value problems and stochastic boundary-value problems are of increasing interest for mathematicians because their theoretical foundation is well developed and new applications are discovered [17]. Many interesting numerical methods for approximating the solution of stochastic initial-value problem have recently been developed, analyzed and tested (see [11]). However, numerical approximation of stochastic boundary-value problems has not been researched thoroughly so far [13] [15] [16].

In this thesis, we describe, analyze and compare the investigation of finite difference methods for numerical solution of stochastic parabolic partial differential equations driven by white noise.

5.2 Stochastic Parabolic Partial Differential Equation :

Consider one-dimensional stochastic parabolic partial differential equation of the form

$$\frac{\partial u (t, x)}{\partial t} - \frac{\partial^2 u (t, x)}{\partial x^2} + bu (t, x) = \frac{\partial^2 w (t, x)}{\partial t \, \partial x} + g (t, x), \quad t > 0 \qquad \dots (5.2)$$

$$u (0, x) = u_0 (x), \quad 0 \leq x \leq 1$$

$$u (t, 0) = u (t, 1) = 0, \quad t \geq 0$$

where $\dfrac{\partial^2 w}{\partial t\, \partial x}$ denotes the mixed second-order derivative of the Brownian sheet and b is a constant.

Here W (t, x) = Time-space wiener process or Brownian motion or Brownian sheet

$$= \sum_{j=1}^{\infty} \Upsilon_j^{1/2}\, e_j(x) \cdot \beta_j(t)$$

$$\text{and operator, } A = -\frac{\partial^2}{\partial x}$$

$$D(A) = H^2 \cap H_0^1$$

$$= \{u \mid u', u'' \in L_2(0, 1),\ u(0) = u(1) = 0\}$$

$$L_2(0, 1) = \left\{ f : \int_0^1 f^2\, dx < \infty \right\}$$

$$(f, g) = \int_0^1 f \cdot g\, dx \qquad \text{… (Hilbert space)}$$

$$A \cdot e_j = \lambda_j \cdot e_j$$

$$e_j = \text{eigen function for operator A}$$

$$= \sqrt{2}\, \sin(j\pi x)$$

$$\lambda_j = j^2\pi^2 \qquad \text{… (eigen value)}$$

What is $\beta_j(t)$?

It is a Brownian motion

$j = 1, 2, \ldots$

Υ_j is some positive number > 0

Usually the required condition is

$$\sum_{j=1}^{\infty} \Upsilon_j < \infty$$

Further $\dfrac{\partial^2 w(t, x)}{\partial t \cdot \partial x}$ does not exist as $W(t, x)$ is not differentiable with respect to t. Brownian sheet depends upon different eigen functions ([1], [2]).

5.2.1 Mild Solution :

The weak formulation of equation (5.2) [26] has the form

$$\int_0^1 u(t, x)\, \phi(x)\, dx - \int_0^t \int_0^1 u(s, x) \frac{d^2\phi}{dx^2}(x)\, dx\, ds + \int_0^t \int_0^1 bu(s, x)\, \phi(x)\, dx\, ds$$

$$= \int_0^1 u_0(x)\, \phi(x)\, dx + \int_0^t \int_0^1 \phi(x)\, dw(s, x) + \int_0^t \int_0^1 g(s, x)\, \phi(x)\, dx\, ds \qquad \ldots (5.3)$$

for $\phi \in c^2[0, 1] \cap c_0[0, 1]$

5.2.2 Strong Solution :

The strong solution (the integral formulation) of equation (5.2) [26] has the form

$$u(t, x) + \int_0^t \int_0^1 G_{t-s}(x, y)\, bu(s, y)\, dy\, ds = \int_0^1 G_t(x, y)\, u_0(y)\, dy$$

$$+ \int_0^t \int_0^1 G_{t-s}(x, y)\, dw(s, y) + \int_0^t \int_0^1 G_{t-s}(x, y)\, g(s, y)\, dy\, ds \qquad \dots (5.4)$$

where $G_t(x, y) = 2 \sum_{n=1}^{\infty} \sin(n\pi x) \cdot \sin(n\pi y) \cdot e^{-(n\pi)^2 t}$ is the fundamental solution of

$$v_t(t, x) - v_{xx}(t, x) = 0$$

$$v(0, x) = \phi(x)$$

$v(t, 0) = v(t, 1) = 0$ so that

$$v(t, x) = \int_0^1 G_t(x, y)\, \phi(y)\, dy$$

In the present thesis, the value of b is assumed to be zero.

5.2.3 Existence and Uniqueness of Solution :

The following result is proved by Buckdahn and Pardoux [27] Concerning existence and uniqueness of solutions (5.3) and (5.4) and is produced here for convenience.

Theorem 5.1 :

Let $u_0 \in c_0[0, 1]$ and $g \in L^2_{(loc)}((0, \infty) \times (0, 1))$ then (5.4) has a unique solution $U \in ((0, \infty) \times [0, 1])$ a.s. Furthermore (5.3) and (5.4) are equivalent.

5.3 Approximation of White Noise :

We approximate the white noise by the formula $\frac{\partial^2 W}{\partial t\, \partial x} \approx \frac{\partial^2 \hat{W}}{\partial t\, \partial x}$.

Consider grids of Brownian sheet as below :

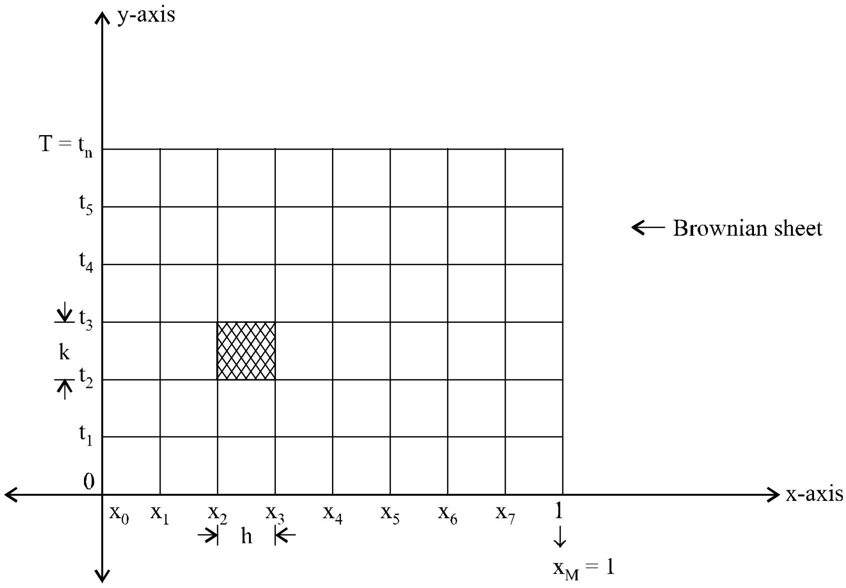

Fig. 5.1 Grids of Brownian sheet

where h = space-step size

k = time-step size

We approximate $\frac{\partial^2 \hat{W}}{\partial t\, \partial x}$ in each small rectangular block ▩ by $\eta_{ij} \sim N(0, 1)$ (Normal distribution). The rectangular block can be thought as

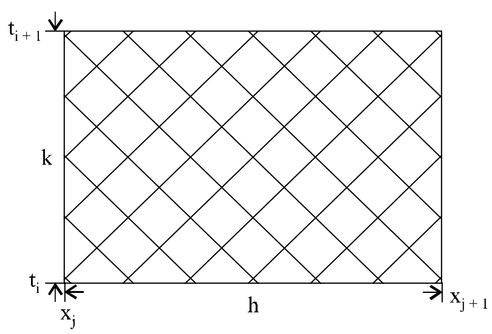

Fig. 5.2

Now consider two dimensional Gaussian white noise $\dfrac{\partial^2 W(t, x)}{\partial t\, \partial x}$ where $W(t, x)$ is Brownian motion on a plane half-strip or a Brownian sheet. Properties of the Brownian sheet are described by Walsh ([26] [27]). Several important properties of the Brownian sheet useful in present thesis are :

(1) If $s = \{(t, x) : a \le t \le b,\ c \le x \le d\}$ is a rectangle, then

$$\int_c^d \int_a^b dW(t, x) = \int_c^d \int_a^b \frac{\partial^2 W(t, x)}{\partial t\, \partial x}\, dt \cdot dx$$

$$= W(s)$$

$$= W(b, d) - W(a, d) - W(b, c) + W(a, c)$$

Where $W(s)$ is Gaussian with zero mean and variance $|s|$ and $|s|$ is the area of s.

(2) If χ_s is the characteristic function of rectangle s, then

$$\int_0^T \int_a^b \chi_s\, dW(t, x) = W(s), \quad \text{for} \quad s \subset (0, T) \times (a, b)$$

(3) If $E\left(\int_0^T \int_a^b f^2(t, x)\, dx\, dt\right) < \infty$ then

$$E\left(\int_0^T \int_a^b f(t, x)\, dW(t, x)\right)^2 = E\left(\int_0^T \int_a^b f^2(t, x)\, dx\, dt\right)$$

Now we define the partition of $[0, T] \times [0, 1]$ by rectangles $[t_i, t_{i+1}] \times [x_j, x_{j+1}]$ for $i = 1,2, \ldots, N$ and $j = 1, 2, \ldots, M$ where $t_i = (i-1)\Delta t$, $x_j = (j-1)\Delta x$, $\Delta t = \dfrac{T}{N}$, and $\Delta x = \dfrac{1}{M}$.

Then, a reasonable approximation on this partition is given by

$$\frac{\partial^2 \hat{W}}{\partial t\, \partial x}(t, x) = \frac{1}{\Delta t \cdot \Delta x} \sum_{i=1}^{N} \sum_{j=1}^{M} \eta_{ij} \sqrt{\Delta t \cdot \Delta x}\; \chi_i(t) \cdot \chi_j(x) \qquad \ldots (5.5)$$

Where χ_i is the characteristic function for the ith sub-interval and

$$\eta_{ij} = \frac{1}{\sqrt{\Delta t \cdot \Delta x}} \int_{t_i}^{t_{i+1}} \int_{x_j}^{x_{j+1}} dW(t, x)$$

that is

$\eta_{ij} \approx N(0, 1)$ and this is a well defined function. Hence it is possible to approximate L.H.S. of the equation (5.5).

$\dfrac{\partial^2 \hat{W}}{\partial t\, \partial x}$ is a piecewise constant on $[0, 1] \times [0, T]$.

$$\therefore \quad \frac{\partial^2 \hat{W}}{\partial t\, \partial x} \in L^2((0, 1) \times (0, T))$$

Now we will approximate the original equation (5.2) as follow :

$$\hat{U}_t - \hat{U}_{xx} = \frac{\partial^2 \hat{W}}{\partial t\, \partial x}, \quad 0 < x < 1,\ t > 0 \qquad \ldots (5.6)$$

$$\hat{U}(t, x) = 0 \quad x = 0,\ x = 1,\ t > 0$$

$$\hat{U}(0, x) = u_0(x)$$

Finally we have to solve above equation (5.6) numerically by using finite difference methods.

Consider $f = \left(\frac{\partial^2 \hat{W}}{\partial t\, \partial x}\right)$. Then we approximate it

Using (i) Forward Euler method

(ii) Backward Euler method

(iii) Crank-Nicolson method.

Next, we shall see the discretization of time t. We write

$$\hat{U}_t\,(t_n, x_j) \approx \frac{U_j^n - U_j^{n-1}}{k}$$

Note that $\underset{\substack{\uparrow \\ \text{Approximate} \\ \text{solution}}}{U_j^n} \approx \underset{\substack{\uparrow \\ \text{Exact} \\ \text{solution}}}{\hat{U}\,(t_n, x_j)}$

and $\hat{U}_{xx}\,(t_n, x_j) \approx \frac{U_{j-1}^n - 2U_j^n + U_{j+1}^n}{h^2}$

where t_n = constant

Then R.H.S. of equation (5.6) is

$$= \frac{\partial^2 \hat{W}}{\partial t\, \partial x}\,(t_n, x_j)$$

$$= \frac{1}{\sqrt{kh}} \cdot \eta_{nj}, \quad \eta_{nj} \sim N\,(0, 1)$$

To generate η_{nj}, we use Matlab. The code function is **'randn'**.

(a) For forward Euler approximation, we have

$$\frac{U_j^n - U_j^{n-1}}{k} - \frac{U_{j-1}^n - 2U_j^n + U_{j+1}^n}{h^2} = \frac{1}{\sqrt{kh}}\,\eta_{nj}$$

This gives the linear system of equations which can be written using matrix notation as below :

$$\begin{bmatrix} 1+\frac{k}{h^2} & -\frac{k}{h^2} & 0 & \cdots & \cdots \\ -\frac{k}{h^2} & 1+\frac{k}{h^2} & -\frac{k}{h^2} & \cdots & \cdots \\ 0 & -\frac{k}{h^2} & 1+\frac{k}{h^2} & -\frac{k}{h^2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & -\frac{k}{h^2} \\ \cdots & \cdots & \cdots & -\frac{k}{h^2} & 1+\frac{k}{h^2} \end{bmatrix}_{(M-1)\,(M-1)} \begin{bmatrix} U_1^n \\ U_2^n \\ \vdots \\ U_{M-1}^n \end{bmatrix}$$

$$= \frac{k}{h^2}\begin{bmatrix} U_1^{n-1} \\ U_2^{n-1} \\ \vdots \\ U_{M-1}^{n-1} \end{bmatrix} + \frac{k}{\sqrt{hk}}\begin{bmatrix} \eta_{n1} \\ \eta_{n2} \\ \vdots \\ \eta_{nM} \end{bmatrix}$$

This in brief can be written as

$$A_n\, U^n = f^n, \quad n = 0, 1, 2, \ldots$$

u^0 is the initial value. This in turn gives u^1 which in turn gives u^2 and so forth.

Thus, $U^0 \to U^1 \to U^2 \to \dots$ (See Fig. 5.3)

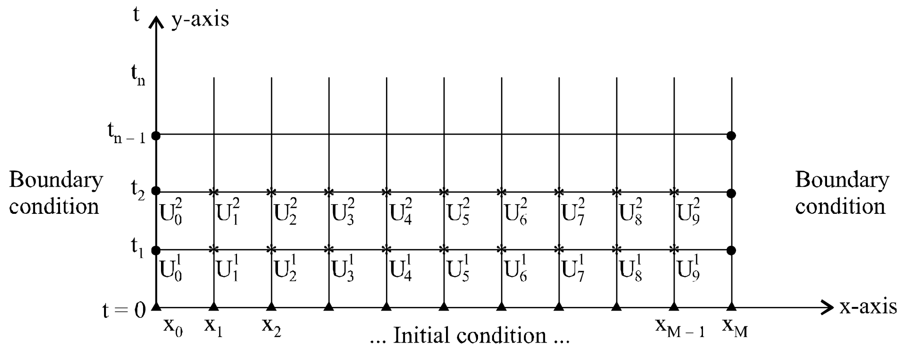

Fig. 5.3

- The solid black circles indicate the location of the (known) boundary values.

- The solid black triangles indicate the location of the (known) initial values.

- The solid black stars indicate the position of the interior points where the finite difference approximation is computed.

We solve this using Matlab.

In section 5.4 we will prove that

$$\{E\,\|U^N - u^N\|^2\}^{1/2} \leq c\,(k^{1/4} + h^{1/2})$$

This can be checked for its validity as below. First we check that

$$(E\,\|U^N - u^N\|^2)^{1/2} \leq ck^{1/4}$$

Here let

$$(E\,\|U^N - u^N\|^2)^{1/2} = e_N$$

Thus $e_N \leq ck^{1/4}$

where h is as small as $h = 2^{-10}$. The error e_n is dominated by k.

Taking logarithm on both sides of this inequality, we get

$$\log(e_n) \le \log(ck^{1/4})$$

$$\le \log c + \frac{1}{4}\log k$$

So this relation is of the form $y = mx + c$, a straight line relationship.

We could plot this function by taking $k = \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \ldots$ and finding e_n corresponding to each value of k. The graph is as shown below in Fig. 5.4.

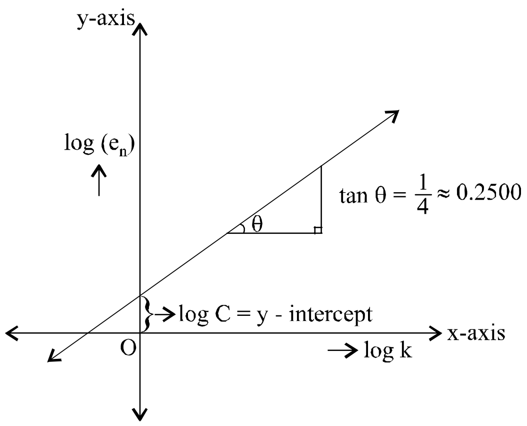

Fig. 5.4

Thus, we have

(a) <u>Original equation</u> :

$$u_t - u_{xx} = \frac{\partial^2 W(t,x)}{\partial t\,\partial x}$$

$$u(t,0) = u(t,1) = 0$$

$$u(0,x) = u_0(x)$$

(b) Approximation equation :

$$\hat{U}_t - \hat{U}_{xx} = \frac{\partial^2 \hat{W}(t, x)}{\partial t\, \partial x}$$

$$\hat{U}(t, 0) = \hat{U}(t, 1) = 0$$

$$\hat{U}(0, x) = U_0(x)$$

(c) Numerical approximation :

$$\hat{U}_t(t_n, x_j) \approx \frac{U_j^n - U_j^{n-1}}{k}$$

$$\hat{U}_{xx}(t_n, x_j) \approx \frac{U_{j-1}^n - 2U_j^n + U_{j+1}^n}{h^2}$$

By formula of heat equation, we have

$$u(t, x) = \int_0^t G(t, x, y)\, u_0(y)\, dy + \int_0^t \int_0^1 G(t - s, x, y)\, dW(s, y) \qquad \dots (5.7)$$

The second integral is called stochastic integral where $G(t, x, y)$ is the Green function which is given by

$$G(t, x, y) = 2 \sum_{j=1}^{\infty} \sin(j\pi x) \sin(j\pi y) \cdot e^{-(j\pi)^2 t}$$

and

$$\hat{U}(t, x) = \int_0^t G(t, x, y)\, u_0(y)\, dy + \int_0^t \int_0^1 G(t-s, x, y)\, d\hat{W}(s, y)$$

$$= \int_0^t G(t, x, y)\, u_0(y)\, dy + \int_0^t \int_0^1 G(t-s, x, y)\, \frac{\partial^2 \hat{W}(s, y)}{\partial s\, \partial y}\, ds\, dy \quad \dots (5.8)$$

The second integral is called Riemann integral.

5.4 Error Estimates :

To obtain error estimation we have to take difference of above equations (5.7) and (5.8). We will discuss the following theorem :

Theorem 5.2 : Let U and $\hat{U}$ be the solution of (5.7) and (5.8) respectively. Then, we have

$$E \int_0^T \int_0^1 \left(U(t, x) - \hat{U}(t, x)\right)^2 dt\, dx \le ck^{1/2} + ch$$

OR

$$\left[E \int_0^T \int_0^1 \left(U - \hat{U}\right)^2 dt \cdot dx \right]^{1/2} \le ck^{1/4} + ch^{1/2}$$

Before we give proof of this theorem, we shall discuss some useful lemmas and theorem necessary for this proof.

Lemma 1 : Show that

$$\sum_{n=1}^{\infty} e^{-2(n\pi)^2 k} (n\pi)^2 k^2 \le ck^{1/2}$$

Proof :

Substitute $n\pi = x$ in this lemma, then, we get

$$\sum_{n=1}^{\infty} e^{-2(n\pi)^2 k} (n\pi)^2 k^2 = \sum_{n=1}^{\infty} e^{-2x^2 k} x^2 k^2$$

$$= \int_1^{\infty} e^{-2x^2 k} x^2 k^2 \, dx \qquad \text{… Expressing sum as integral}$$

By change of variable method,

Let $xk^{1/2} = y \qquad \therefore \quad x^2 k = y^2$

and $dx \cdot k^{1/2} = dy \;\Rightarrow\; dx = k^{-1/2} dy$

Then the above integral reduces to

$$= \int_1^{\infty} e^{-2y^2} \frac{y^2}{k} \cdot k^2 k^{-1/2} dy$$

$$= \underbrace{\int_1^{\infty} e^{-2y^2} y^2 dy}_{\Downarrow} \cdot k^{1/2}$$

This integral is convergent and hence bounded.

$$\leq ck^{1/2}$$

Hence the proof.

Lemma 2 : Show that

$$\sum_{n=1}^{\infty} \frac{1 - e^{-2(n\pi)^2 k}}{(n\pi)^2} \leq ck^{1/2}$$

Proof : Here also substitute $n\pi = x$ and we get

$$\sum_{n=1}^{\infty} \frac{1 - e^{-2(n\pi)^2 k}}{(n\pi)^2} = \sum_{n=1}^{\infty} \frac{1 - e^{-2x^2 k}}{x^2}$$

$$= \int_1^{\infty} \frac{1 - e^{-2x^2 k}}{x^2}\, dx \qquad \text{… Expressing the sum as integral}$$

By change of variable method

Let $xk^{1/2} = y \quad \Rightarrow \quad y^2 = x^2 k$

$\therefore \quad dx \times k^{1/2} = dy \quad \Rightarrow \quad dx = k^{-1/2}\, dy$

Then the above integral reduces to

$$= \int_1^{\infty} \frac{1 - e^{-2y^2}}{y^2} \cdot k \cdot dy \cdot k^{-1/2}$$

$$= \underbrace{\int_1^{\infty} \frac{1 - e^{-2y^2}}{y^2}\, dy}_{\text{This integral is convergent and hence bounded.}} \cdot k^{1/2}$$

$$\leq ck^{1/2}$$

Hence the proof.

Lemma 3 : Show that

$$\sum_{n=1}^{\infty} e^{-2(n\pi)^2 k} \leq ck^{-1/2}$$

Proof : Upon substituting $n\pi = x$ in above summation, we get

$$\sum_{n=1}^{\infty} e^{-2(n\pi)^2 k} = \sum_{n=1}^{\infty} e^{-2x^2 k}$$

$$= \int_1^{\infty} e^{-2x^2 k}\, dx \qquad \text{… Expressing the sum as integral}$$

By change of variable method,

Let $xk^{1/2} = y \Rightarrow x^2 k = y^2$

and $dx \cdot k^{1/2} = dy \Rightarrow dx = k^{-1/2}\, dy$

Then the above integral reduces to

$$= \int_1^{\infty} e^{-2y^2} \cdot k^{-1/2} \cdot dy$$

$$= \underbrace{\int_1^{\infty} e^{-2y^2}\, dy}_{\text{This integral is convergent and hence bounded}} \cdot k^{-1/2}$$

$$\leq ck^{-1/2}$$

Hence the proof.

In addition to above lemmas, the following theorem is essential for the proof of theorem 5.2.

Theorem 5.3 : Let $f \in L^2([0, T) \times [0, 1])$. Then, we have

$$E \left| \int_0^T \int_0^1 f(s, y)\, d\hat{W}(s, y) \right|^2 \leq E \int_0^T \int_0^1 f^2(s, y)\, ds\, dy$$

Proof :

$$\text{L.H.S.} = E \left| \int_0^T \int_0^1 f(s, y)\, d\hat{W}(s, y) \right|^2$$

$\uparrow$ This is piece-wise constant

$$= E \left| \sum_{j=1}^{N} \sum_{i=1}^{M} \int_{t_j}^{t_{j+1}} \int_{x_i}^{x_{i+1}} f(x, y) \frac{\int_{t_j}^{t_{j+1}} \int_{x_i}^{x_{i+1}} dW(r, z)}{kh} \, ds\, dy \right|^2$$

where dW (r, z) is normally distributed.

$$= E \left| \sum_{j=1}^{N} \sum_{i=1}^{M} \int_{t_j}^{t_{j+1}} \int_{x_i}^{x_{i+1}} \left[\frac{1}{kh} \int_{t_j}^{t_{j+1}} \int_{x_i}^{x_{i+1}} f(s, y)\, ds\, dy \right] dW(r, z) \right|^2$$

where dW (r, z) is original Brownian motion.

Again

$$E \left\| \int_0^T \int_0^1 f(s, y)\, dW(s, y) \right\|^2 = E \int_0^T \int_0^1 f^2(s, y)\, ds\, dy$$

$$\therefore \quad \text{L.H.S.} = E \sum_{j=1}^{N} \sum_{i=1}^{M} \int_{t_j}^{t_{j+1}} \int_{x_i}^{x_{i+1}} \left[\frac{1}{kh} \int_{t_j}^{t_{j+1}} \int_{x_i}^{x_{i+1}} 1 \cdot f(s, y)\, ds\, dy \right]^2 dr\, dz$$

where g = 1

Now by Cauchy-Swartz inequality, we have

$$\text{L.H.S.} \leq E \sum_{j=1}^{N} \sum_{i=1}^{M} \int_{t_j}^{t_{j+1}} \int_{x_i}^{x_{i+1}}$$

$$\frac{1}{(kh)^2} \left(\int_{t_j}^{t_{j+1}} \int_{x_i}^{x_{i+1}} f^2(s, y)\, ds\, dy \right) \cdot \underbrace{\left(\int_{t_j}^{t_{j+1}} \int_{x_i}^{x_{i+1}} 1^2\, ds\, dy \right)}_{\Downarrow \atop \text{This is equal to kh.}} dr\, dz$$

$$= E \sum_{j=1}^{N} \sum_{i=1}^{M} \int_{t_j}^{t_{j+1}} \int_{x_i}^{x_{i+1}} \frac{1}{kh} \int_{t_j}^{t_{j+1}} \int_{x_i}^{x_{i+1}} f^2(s, y)\, ds\, dy\, dr\, dz$$

$$= E \int_0^T \int_0^1 f^2(s, y)\, ds\, dy$$

$$= \text{R.H.S.}$$

Hence the proof.

Proof of theorem 5.2 :

Consider

$$E \int_0^T \int_0^1 \left(U(t, x) - \hat{U}(t, x) \right)^2 dt\, dx$$

$$= E \int_0^T \int_0^1 \left[\int_0^T \int_0^1 G(t-s, x, y)\, dW(s, y) - \int_0^t \int_0^1 G(t-s, x, y)\, d\hat{W}(s, y) \right]^2 dt\, dx$$

Now we will follow discretization for t,

$$= E \sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} \int_{0}^{1} \Big[\underbrace{\int_{0}^{t} \int_{0}^{1} G(t-s, x, y)\, dW(s, y) - \int_{0}^{t_j} \int_{0}^{1} G(t_j - s, x, y)\, dW(s, y)}_{\mathrm{I}}$$

$$+ \underbrace{\int_{0}^{t_j} \int_{0}^{1} G(t_j - s, x, y)\, dW(s, y) - \int_{0}^{t_j} \int_{0}^{1} G(t_j - s, x, y)\, d\hat{W}(s, y)}_{\mathrm{II}}$$

$$+ \underbrace{\int_{0}^{t_j} \int_{0}^{1} G(t_j - s, x, y)\, d\hat{W}(s, y) - \int_{0}^{t} \int_{0}^{1} G(t-s, x, y)\, d\hat{W}(s, y)}_{\mathrm{III}} \Big]^2 dt\, dx$$

$$= \mathrm{I} + \mathrm{II} + \mathrm{III}$$

Now consider

$$\mathrm{I} = E \sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} \int_{0}^{1} \Big[\int_{0}^{t_j} \int_{0}^{1} G(t-s, x, y) - G(t_j - s, x, y)\, dW(s, y) \Big]^2 dt\, dx$$

$$+ E \sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} \int_{0}^{1} \int_{t_j}^{t} \int_{0}^{1} [G(t-s, x, y)\, dW(s, y)]^2 dt\, dx$$

$$= \mathrm{I}_1 + \mathrm{I}_2$$

where by Isometry property

$$\mathrm{I}_1 = E \sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} \int_{0}^{1} \int_{0}^{t_j} \int_{0}^{1} [G(t-s, x, y) - G(t_j - s, x, y)]^2 dy\, ds\, dx\, dt$$

We use green function to estimate I_1.

Thus, here, we have

$$G(t, x, y) = \sum_{n=1}^{\infty} \sin n\pi x \cdot \sin n\pi y \cdot e^{-(n\pi)^2 t}$$

Then

$$I_1 = E \sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} \int_0^1 \int_0^{t_j} \int_0^1$$

$$\left[\sum_{n=1}^{\infty} \sin(n\pi x) \sin(n\pi y) \{e^{-(n\pi)^2 (t-s)} - e^{-(n\pi)^2 (t_j - s)}\} \right]^2 dy\, ds\, dx\, dt$$

Where using $\int_0^1 \sin(n\pi x) \cdot \sin(m\pi x)\, dx = \begin{cases} 1 & \text{for } m = n \\ 0 & \text{for } m \neq n \end{cases}$

and similarly $\int_0^1 \sin(n\pi y) \cdot \sin(m\pi x)\, dx = \begin{cases} 1 & \text{for } m = n \\ 0 & \text{for } m \neq n \end{cases}$

we get

$$\therefore \quad I_1 = E \sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} \int_0^{t_j} \sum_{n=1}^{\infty} (e^{-(n\pi)^2 (t-s)} - e^{-(n\pi)^2 (t_j - s)})^2\, ds\, dt$$

$$= E \sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} \sum_{n=1}^{\infty} \int_0^{t_j} e^{-2(n\pi)^2 (t-s)} \cdot (1 - e^{-(n\pi)^2 (t_j - t)})^2\, ds\, dt$$

$$= E \sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} \sum_{n=1}^{\infty} (1 - e^{-(n\pi)^2 (t_j - t)})^2 \cdot \frac{e^{-2(n\pi)^2 (t - t_j)} - e^{-2(n\pi)^2 t}}{2(n\pi)^2} \, dt$$

$$= E \sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} \sum_{n=1}^{\infty} [\; e^{-2(n\pi)^2 (t_j - t)} (e^{(n\pi)^2 (t_j - t)} - 1)^2 \cdot \frac{e^{-2(n\pi)^2 (t - t_j)} - e^{-2(n\pi)^2 t}}{2(n\pi)^2} \;] \, dt$$

$$= E \sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} \sum_{n=1}^{\infty} (1 - e^{-(n\pi)^2 (t - t_j)})^2 \cdot \frac{1 - e^{-2(n\pi)^2 t_j}}{2(n\pi)^2} \, dt$$

$$\leq E \sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} \sum_{n=1}^{\infty} (1 - e^{-(n\pi)^2 (t - t_j)^2}) \cdot \frac{1}{2(n\pi)^2} \, dt$$

$$\leq E \sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} \sum_{n=1}^{\infty} (1 - e^{-(n\pi)^2 k})^2 \cdot \frac{1}{2(n\pi)^2} \, dt \qquad \ldots (A)$$

$$\leq k^{1/2}$$

Why less than $k^{1/2}$? It can be sown as below.

From step (A),

$$\leq T \sum_{n=1}^{\infty} (1 - e^{-(n\pi)^2 k})^2 \cdot \frac{1}{2(n\pi)^2}$$

$$\leq T \int_{0}^{\infty} (1 - e^{-(y\pi)^2 k})^2 \cdot \frac{1}{2(y\pi)^2} \, dy \qquad \ldots \text{Taking } y = n$$

$$\leq c \int_0^\infty (1 - e^{-z^2 k})^2 \cdot \frac{1}{2z^2} \, dz \qquad \ldots \text{ where } y\pi = z$$

(Taking constant terms in c) $\qquad \therefore \; dy \cdot \pi = dz$

Finally let $z\sqrt{k} = X, \quad \therefore \; dz\sqrt{k} = dx$

and $z^2k = x^2$

$$\therefore \quad I_1 \leq c \int_0^\infty (1 - e^{-x^2})^2 \cdot \frac{k}{x^2} \cdot \frac{dx}{\sqrt{k}}$$

$$\leq ck^{1/2} \int_0^\infty (1 - e^{-x^2})^2 \cdot \frac{1}{x^2} \, dx$$

$$\leq ck^{1/2} \qquad \ldots \text{ As the integral is convergent and hence bounded}$$

Thus proof for I completes.

Next, consider stochastic integral-II.

$$II = E \sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} \int_0^1 \{ [\int_0^{t_j} \int_0^1 G(t_j - s, x, y) \, \underset{\substack{\uparrow \\ \text{Time-space wiener process}}}{dW(s, y)} -$$

$$\int_0^{t_j} \int_0^1 G(t_j - s, x, y) \, \underset{\substack{\uparrow \\ \text{Piece-wise constant approximation}}}{d\hat{W}(s, y)} \;] \}^2 \, dx \, dt$$

$$= E \sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} \int_{0}^{1} \{ \sum_{l=1}^{j-1} \sum_{i=1}^{M} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} G(t_j - r, x, z)\, dW(s, y)$$

$$- \sum_{l=1}^{j-1} \sum_{i=1}^{M} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} G(t_j - s, x, y)\, d\hat{W}(s, y) \}^2 \, dx\, dt$$

Here by definition

$$\int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} G(t_j - s, x, y)\, d\hat{W}(s, y) = \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} G(t_j - s, x, y)\, \eta_{li} \cdot \frac{1}{kh}\, dx\, dt$$

Where $\eta_{li} \sim N(0, 1)$ and also we can write

$$\eta_{li} = \frac{1}{kh} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} dW(r, z)$$

Then

$$II = E \sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} \int_{0}^{1} \{ \sum_{l=1}^{j-1} \sum_{i=1}^{M} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} [G(t_j - r, x, z) -$$

$$\int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} G(t_j - s, x, y)\, dx\, dy]\, dw(r, z) \}^2 \, dx\, dt$$

$$\frac{1}{kh} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} [G(t_j - r, x, z) - G(t_j - s, x, y)]\, ds\, dy$$

Putting together, we get

$$\text{II} = E \sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} \int_{0}^{1} \Big\{ \sum_{l=1}^{j-1} \sum_{i=1}^{M} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} \frac{1}{kh} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} [G(t_j - r, x, z)$$

$$- G(t_j - s, x, y)\, ds\, dy\, dw(r, z) \Big\}^2 dx\, dt$$

By isometry property, we get

$$= E \sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} \int_{0}^{1} \sum_{l=1}^{j-1} \sum_{i=1}^{M} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}}$$

$$\Big[\frac{1}{kh} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} \{G(t_j - r, x, z) - G(t_j - s, x, y)\}\, ds\, dy \Big]^2 dr\, dz\, dx\, dt$$

Note : Cauchy-Schwartz inequality

$$\left| \int_a^b f \cdot g\, dx \right|^2 \le \left(\int_a^b f^2\, dx \right) \left(\int_a^b g^2\, dx \right)$$

In present case,

$$f = G(t_j - r, x, z) - G(t_j - s, x, y) \text{ and } g = 1$$

Thus, we have

$$\int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} [\underbrace{\{G(t_j - r, x, z) - G(t_j - s, x, y)\}}_{f} \cdot \underbrace{1}_{g = 1}\, dx\, dy]^2$$

$$\le \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} [\{G(t_j - r, x, y) - G(t_j - s, x, y)\}^2\, ds\, dy \cdot \underbrace{\int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} 1 \cdot ds\, dy}_{\text{This is equal to area} = kh}$$

Then

$$\text{II} = E \sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} \int_{0}^{1} \sum_{l=1}^{j-1} \sum_{i=1}^{M} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}}$$

$$\left\{ \frac{1}{kh} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} [G(t_j - r, x, z) - G(t_j - s, x, y)]^2 \, ds \, dy \right\} dr \, dz \, dx \, dt \quad \dots \text{(B)}$$

Next, let us estimate Green function. By definition of Green function, we have

$$G(t_j - r, x, z) - G(t_j - s, x, y)$$

$$= 2 \sum_{n=1}^{\infty} \sin n\pi x \, [\sin(n\pi z) \cdot e^{-(n\pi)^2 (t_j - r)} - \sin(n\pi y) \cdot e^{-(n\pi)^2 (t_j - s)}]$$

$$\therefore \quad [G(t_j - r, x, z) - G(t_j - s, x, y)]^2$$

$$= \left[2 \sum_{n=1}^{\infty} \sin(n\pi x) \{\sin(n\pi z) \cdot e^{-(n\pi)^2 (t_j - r)} - \sin(n\pi y) \cdot e^{-(n\pi)^2 (t_j - s)}\} \right]^2$$

Now using the property

$$2 \int_0^1 \sin(n\pi x) \cdot \sin(m\pi x) \, dx = 1 \quad \dots \text{for } m = n$$

$$= 0 \quad \dots \text{for } m \neq n$$

Then, we get

$$\int_0^1 [G(t_j - r, x, z) - G(t_j - s, x, y)]^2 \, dx$$

$$= 2 \sum_{n=1}^{\infty} \{\sin n\pi z \cdot e^{-(n\pi)^2 (t_j - r)} - \sin n\pi y \cdot e^{-(n\pi)^2 (t_j - s)}\}^2$$

Using this result into expression (B), we get

$$\text{II} = E \sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} \sum_{l=1}^{j-1} \sum_{i=1}^{M} \Big\{ \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} \Big[\frac{1}{kh} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}}$$

$$\Big(2 \sum_{n=1}^{\infty} (\sin n\pi z \cdot e^{-(n\pi)^2 (t_j - r)} - \sin n\pi y \cdot e^{-(n\pi)^2 (t_j - s)})^2\Big) \Big]\, dy\, ds\, dz\, dr \Big\}\, dt$$

Taking t_j common term outside

$$= E \sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} \Big\{ \sum_{l=1}^{j-1} \sum_{i=1}^{M} \Big\{ \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} \Big[\frac{1}{kh} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} 2 \sum_{n=1}^{\infty} e^{-2 (n\pi)^2 t_j}$$

$$\underbrace{(\sin n\pi z)\, e^{(n\pi)^2 r} - \sin n\pi y\, e^{(n\pi)^2 s}}_{\Downarrow})^2\, dy\, ds \Big] \Big\}\, dz\, dr \Big\}\, dt$$

Note that

$$\sin (n\pi z)\, e^{(n\pi)^2 r} - \sin (n\pi y)\, e^{(n\pi)^2 s}$$

$$= (\sin (n\pi z) - \sin (n\pi y))\, e^{(n\pi)^2 r} + \sin (n\pi y)\, (e^{(n\pi)^2 r} - e^{(n\pi)^2 s})$$

$$\le E \sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} \sum_{l=1}^{j-1} \sum_{i=1}^{M} \Big\{ \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} \frac{1}{kh} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} \sum_{n=1}^{\infty} \Big[e^{-(n\pi)^2 t_j}$$

$$(\sin n\pi z - \sin n\pi y)^2\, e^{2 (n\pi)^2 r}\, dy\, ds \Big]\, dz\, dr \Big\}\, dt$$

$$+ E \sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} \sum_{l=1}^{j-1} \sum_{i=1}^{M} \Big\{ \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} \Big[\frac{1}{kh} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}}$$

$$\sum_{n=1}^{\infty} e^{-(n\pi)^2 t_j} \cdot \sin^2 n\pi y \left(e^{(n\pi)^2 r} - e^{(n\pi)^2 s}\right)^2 dy\, ds \;\Big]\, dz\, dr\Big\}\, dt$$

$$= II_1 + II_2$$

Consider

$$II_2 = E \sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} \sum_{l=1}^{j-1} \sum_{i=1}^{M} \Big\{ \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} \frac{1}{kh} \Big[\int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}}$$

$$2 \sum_{n=1}^{\infty} e^{-2(n\pi)^2 t_j} \cdot \underbrace{\sin^2 (\pi y)}_{\text{This quantity} \leq 1} \left(e^{(n\pi)^2 r} - e^{(n\pi)^2 s}\right)^2 dy\, ds \;\Big]\, dz\, dr \Big\}\, dt$$

y and z are not involved. $\therefore$ their integral = 1. They disappear.

$$= E \sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} \sum_{l=1}^{j-1} \int_{t_l}^{t_{l+1}} \frac{1}{k} \underbrace{\int_{t_l}^{t_{l+1}} 2 \sum_{n=1}^{\infty} e^{-2(n\pi)^2 t_j} \left(e^{(n\pi)^2 r} - e^{(n\pi)^2 s}\right)^2 ds}_{\text{Expressing this integral as sum of adjacent integrals}}\, dr\, dt$$

$$= E \sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} \sum_{l=1}^{j-1} \int_{t_l}^{t_{l+1}} \frac{1}{k} \Big[\int_{t_l}^{r} 2 \sum_{n=1}^{\infty} e^{-2(n\pi)^2 t_j} \left(e^{(n\pi)^2 r} - e^{(n\pi)^2 s}\right)^2 ds \;\Big]\, dr\, dt$$

$$+ E \sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} \sum_{l=1}^{j-1} \int_{t_l}^{t_{l+1}} \frac{1}{k} \Big[\int_{r}^{t_{l+1}} 2 \sum_{n=1}^{\infty} e^{-2(n\pi)^2 t_j} \left(e^{(n\pi)^2 r} - e^{(n\pi)^2 s}\right)^2 \;\Big]\, dr\, dt$$

$$= II_{21} + II_{22}$$

Now

$$II_{21} = E\sum_{j=1}^{N}\int_{t_j}^{t_{j+1}}\sum_{l=1}^{j-1}\int_{t_l}^{t_{l+1}}\frac{1}{k}\left[\int_{t_l}^{r} 2\sum_{n=1}^{\infty} e^{-2(n\pi)^2(t_j-r)}\left(1-e^{-(n\pi)^2(r-s)}\right)^2 ds\right] dr\,dt$$

Here $\overbrace{\underset{t_l}{+}\!-\!\underset{s}{+}\!-\!\underset{r}{+}\!-\!\underset{t_{l+1}}{+}}^{k}$ $\therefore$ $r-s<k$ = time-step

$\therefore$ $\left(1-e^{-(n\pi)^2 k}\right)^2$ = constant and can be moved outside

$$= E\sum_{j=1}^{N}\int_{t_j}^{t_{j+1}}\sum_{l=1}^{j-1}\frac{1}{k}\left[\int_{t_l}^{t_{l+1}}\int_{t_l}^{r} 2\sum_{n=1}^{\infty} e^{-(n\pi)^2(t_j-r)} ds\,dr\right] dt\left(1-e^{-(n\pi)^2 k}\right)^2$$

Note that $\int_{t_l}^{t_{l+1}} ds = k$ will get cancelled with $\frac{1}{k}$ and

$t_l \le r \le t_{l+1}$ so we can write

$$II_{21} \le E\sum_{j=1}^{N}\int_{t_j}^{t_{j+1}}\sum_{l=1}^{j-1}\frac{1}{k}\left[k\int_{t_l}^{t_{l+1}} 2\sum_{n=1}^{\infty} e^{-2(n\pi)^2(t_j-r)} dr\right] dt\cdot\left(1-e^{-(n\pi)^2 k}\right)^2$$

$$= E\sum_{j=1}^{N}\int_{t_j}^{t_{j+1}}\underbrace{\sum_{l=1}^{j-1}\int_{t_l}^{t_{l+1}}}_{\Downarrow} 2\sum_{n=1}^{\infty} e^{-2(n\pi)^2(t_j-r)} dr\,dt\left(1-e^{-(n\pi)^2 k}\right)^2$$

Combining two big integrals

$$= E\sum_{j=1}^{N}\int_{t_j}^{t_{j+1}}\left[\int_{0}^{t_j} 2\underset{\uparrow}{\sum_{n=1}^{\infty}} e^{-2(n\pi)^2(t_j-r)} dr\right] dt\cdot\left(1-e^{-(n\pi)^2 k}\right)^2$$

To estimate it we divide into sum of two integrals

$$= E\sum_{j=1}^{N}\int_{t_j}^{t_{j+1}}\left[\int_{0}^{t_{j-1}} 2\sum_{n=1}^{\infty} e^{-2(n\pi)^2(t_j-r)}\,dr\ \right]dt\cdot\left(1-e^{-(n\pi)^2 k}\right)^2$$

$$+\, E\sum_{j=1}^{N}\int_{t_j}^{t_{j+1}}\left[\int_{t_{j-1}}^{t_j} 2\sum_{n=1}^{\infty} e^{-2(n\pi)^2(t_j-r)}\,dr\ \right]dt\cdot\left(1-e^{-(n\pi)^2 k}\right)^2$$

We use $1-e^{-x}\le x,\quad x>0$ in first integral

and $1-e^{-x}\ \le\ 1,\quad x>0$ in second integral.

Hence

$$II_{21}\le\ E\sum_{j=1}^{N}\int_{t_j}^{t_{j+1}}\left[\int_{0}^{t_{j-1}} 2\sum_{n=1}^{\infty} e^{-2(n\pi)^2(t_j-r)}\,dr\ \right]dt\,\left((n\pi)^2 k\right)^2$$

$$+\, E\sum_{j=1}^{N}\int_{t_j}^{t_{j+1}}\left[\int_{t_{j-1}}^{t_j} 2\sum_{n=1}^{\infty} e^{-2(n\pi)^2(t_j-r)}\,dr\ \right]dt\cdot 1$$

Integrating w.r.t. x, the integrals yield

$$= E\sum_{j=1}^{N}\int_{t_j}^{t_{j+1}} 2\sum_{n=1}^{\infty}\frac{e^{-2(n\pi)^2 k}-e^{-2(n\pi)^2 t_j}}{2(n\pi)^2}\ \left((n\pi)^2 k\right)^2 dt$$

$$+\, E\sum_{j=1}^{N}\int_{t_j}^{t_{j+1}} 2\sum_{n=1}^{\infty}\frac{1-e^{-2(n\pi)^2 k}}{2(n\pi)^2}\ dt$$

Now $\displaystyle\sum_{j=1}^{N}\int_{t_j}^{t_{j+1}} = T$

$$II_{21} = 2T \sum_{n=1}^{\infty} \frac{e^{-2(n\pi)^2 k} - e^{-2(n\pi)^2 t_j}}{2(n\pi)^2} \left((n\pi)^2 k\right)^2$$

$$+ 2T \sum_{n=1}^{\infty} \frac{1 - e^{-2(n\pi)^2 k}}{2(n\pi)^2}$$

Removing the term $e^{-2(n\pi)^2 t_j}$, we get

$$\leq T \sum_{n=1}^{\infty} e^{-2(n\pi)^2 k} \cdot (n\pi)^2 k^2 + T \sum_{n=1}^{\infty} \frac{1 - e^{-2(n\pi)^2 k}}{(n\pi)^2}$$

Finally, using the lemmas

$$(1) \sum_{n=1}^{\infty} e^{-2(n\pi)^2 k} (n\pi)^2 k^2 \leq ck^{1/2}$$

and $(2) \displaystyle\sum_{n=1}^{\infty} \frac{1 - e^{-2(n\pi)^2 k}}{(n\pi)^2} \leq ck^{1/2}$, we get

$$II_{21} \leq ck^{1/2}$$

Further

$$II_{22} = E \sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} \sum_{l=1}^{j-1} \int_{t_l}^{t_{l+1}} \frac{1}{k} \left[\int_{r}^{t_{l+1}} 2 \sum_{n=1}^{\infty} e^{-2(n\pi)^2 t_j} \left(e^{(n\pi)^2 r} - e^{(n\pi)^2 s}\right)^2 ds \right] dr\, dt$$

$$= E \sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} \sum_{l=1}^{j-1} \int_{t_l}^{t_{l+1}} \frac{1}{k} \left[\int_{r}^{t_{l+1}} 2 \sum_{n=1}^{\infty} e^{-2(n\pi)^2 (t_j - s)} \left(1 - e^{-(n\pi)^2 (s - r)}\right)^2 ds \right] dr\, dt$$

Here $s - r < k$

$$= E\sum_{j=1}^{N}\int_{t_j}^{t_{j+1}}\sum_{l=1}^{j-1}\int_{t_l}^{t_{l+1}}\frac{1}{k}\Big[\int_{r}^{t_{l+1}} 2\sum_{n=1}^{\infty} e^{-2(n\pi)^2(t_j-s)}(1-e^{-(n\pi)^2 k})^2\,ds\Big]\,dr\,dt$$

$\uparrow$

Replacing r by t_l,

$$\le E\sum_{j=1}^{N}\int_{t_j}^{t_{j+1}}\sum_{l=1}^{j-1}\int_{t_l}^{t_{l+1}}\frac{1}{k}\Big[\int_{t_l}^{t_{l+1}} 2\sum_{n=1}^{\infty} e^{-2(n\pi)^2(t_j-s)}(1-e^{-(n\pi)^2 k})^2\,ds\Big]\,dr\,dt$$

$$= E\sum_{j=1}^{N}\int_{t_j}^{t_{j+1}}\sum_{l=1}^{j-1}\frac{1}{k}\Big[\int_{t_l}^{t_{l+1}}\int_{t_l}^{t_{l+1}} 2\sum_{n=1}^{\infty} e^{-2(n\pi)^2(t_j-s)}\,ds\,dr\Big]\,dt\,(1-e^{-(n\pi)^2 k})^2$$

$$\le ck^{1/2} \quad (\text{same as } II_{21})$$

Now let us consider

$$II_1 = E\sum_{j=1}^{N}\int_{t_j}^{t_{j+1}}\sum_{l=1}^{j-1}\sum_{i=1}^{M}\int_{t_l}^{t_{l+1}}\int_{x_i}^{x_{i+1}}\frac{1}{kh}\int_{t_l}^{t_{l+1}}\int_{x_i}^{x_{i+1}}$$

$$2\sum_{n=1}^{\infty} e^{-2(n\pi)^2(t_j-r)}(\sin(n\pi z)-\sin(n\pi y))^2\,dy\,ds\,dz\,dr\,dt$$

Note that $\quad |\sin(n\pi z)-\sin(n\pi y)| \le 1+1$

and $\quad |\sin(n\pi z)-\sin(n\pi y)|^2 \le (n\pi z - n\pi y)^2$

$$\le n\pi(z-y)$$

$$\le n\pi h$$

where $h = |y-z|$

Here $x_i \le y \le x_{i+1} \quad x_i \le z \le x_{i+1}$

$x_{i+1} - x_i = h$, the term $\sum_{l=1}^{j-1}$ disappears in the next step.

Hence, we get

$$II_1 \le \sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} \sum_{i=1}^{M} \Big[\int_0^{t_{j-1}} \int_{x_i}^{x_{i+1}} \frac{1}{kh} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}}$$

$$2 \sum_{n=1}^{\infty} e^{-2(n\pi)^2 (t_j - r)} (n\pi h)^2 \, dy\, ds\, dz\, dr\, dt \Big]$$

$$+ \sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} \sum_{i=1}^{M} \Big[\int_{t_{j-1}}^{t_j} \int_{x_i}^{x_{i+1}} \frac{1}{kh} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}}$$

$$2 \sum_{n=1}^{\infty} e^{-2(n\pi)^2 (t_j - r)} 2^2 \, dy\, ds\, dz\, dr\, dt \Big]$$

$$\le \sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} \int_0^{t_{j-1}} 2 \sum_{n=1}^{\infty} e^{-2(n\pi)^2 (t_j - r)} (n\pi h)^2 \, dr\, dt$$

$$+ \sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} \int_{t_{j-1}}^{t_j} 2 \sum_{n=1}^{\infty} e^{-2(n\pi)^2 (t_j - r)} 4 \, dr\, dt$$

$$= \sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} 2 \sum_{n=1}^{\infty} \frac{1}{2(n\pi)^2} \left(e^{-2(n\pi)^2 k} e^{-2(n\pi)^2 t_j}\right) (n\pi h)^2 \, dt$$

$$+ \sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} 8 \cdot \frac{1 - e^{-2(n\pi)^2 k}}{2(n\pi)^2} \, dt$$

Here $\sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} = T$, constant and dropping the term $e^{-2(n\pi)^2 t_j}$, we get

$$II_1 = T \sum_{n=1}^{\infty} e^{-2(n\pi)^2 k} \cdot h^2 + 8T \sum_{n=1}^{\infty} \frac{1 - e^{-2(n\pi)^2 k}}{2(n\pi)^2}$$

To estimate these two summations we use the lemmas

(1) $\sum_{n=1}^{\infty} e^{-2(n\pi)^2 k} \le ck^{-1/2}$

(2) $\sum_{n=1}^{\infty} \frac{1 - e^{-2(n\pi)^2 k}}{2(n\pi)^2} \le ck^{1/2}$

$$\therefore \quad II_1 \le T\,ck^{-1/2} h^2 + 8T\,ck^{1/2}$$

$$\le ck^{1/2} \qquad \ldots \text{(Assuming } h^2 = k)$$

Finally, let us consider

$$III = E \sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} \int_0^1 \Big[\int_0^t \int_0^1 G(t-s, x, y)\, d\hat{W}(s, y) - \int_0^{t_j} \int_0^1 G(t_j - s, x, y)\, d\hat{W}(s, y) \Big]^2 dx\, dt$$

$$= E \sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} \int_0^1 \Big[\int_0^{t_j} \int_0^1 G(t-s, x, y)\, d\hat{W}(s, y) \underbrace{-\int_0^{t_j} \int_0^1 G(t_j - s, x, y)\, d\hat{W}(s, y)}_{\text{Putting together}}$$

$$+ \int_{t_j}^{t} G(t-s, x, y)\, d\hat{W}(s, y) \Big]^2 dx\, dt$$

$= \mathrm{III}_1 + \mathrm{III}_2$ (Using the property $(a+b)^2 \le 2(a^2+b^2)$)

where

$$\mathrm{III}_1 \le 2\,E \sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} \int_0^1 \left[\int_0^{t_j} \int_0^1 G(t-s,x,y) - G(t_j - s, x, y)\, d\hat{W}(s,y) \right]^2 dx\, dt$$

Now by isometry property

$$E \left| \int_0^T \int_0^1 f(s,y)\, d\hat{W}(s,y) \right|^2 = E \int_0^T \int_0^1 f^2(s,y)\, ds\, dy$$

We can write

$$\mathrm{III}_1 = 2\,E \sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} \int_0^1 \left[\int_0^{t_j} \int_0^1 \{G(t-s,x,y) - G(t_j - s, x, y)\} \right]^2 ds\, dy\, dx\, dt$$

$$\le c k^{1/2} \quad \ldots \text{(See the proof } I_1\text{)}$$

And

$$\mathrm{III}_2 = E \sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} \int_0^1 \left[\int_{t_j}^{t} \{G(t-s,x,y)\, d\hat{W}(s,y) \right]^2 dx\, dt$$

Again using isometry property, we get

$$= E \sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} \int_0^1 \int_{t_j}^{t} G(t-s,x,y)^2\, ds\, dy\, dx\, dt$$

$$\le c k^{1/2} \quad \ldots \text{(See the proof } I_2\text{)}$$

The proof of Theorem 5.2 is complete.

5.5 Numerical Investigation (Stochastic) :

- Test Problem :

In this section, we will consider the numerical simulation for the following problem :

$$u_t - u_{xx} + bu = f + g \qquad t > 0 \qquad \dots (5.9)$$

$$u(0, x) = 10x^2 (1 - x^2) \qquad 0 \le x \le 1$$

$$u(t, 0) = u(t, 1) = 0$$

Here $$f = \frac{\partial^2 W}{\partial t \, \partial x}$$

$$g = 10(1 + b)\, x^2 (1 - x)^2 e^t - 10(2 - 12x + 12x^2)\, e^t$$

$$b = 0.5$$

The approximate solution of problem (5.9) is

$$\hat{u}_t - \hat{u}_{xx} + b\hat{u} = \hat{f} + g \qquad t > 0$$

$$\hat{u}(0, x) = 10x^2 (1 - x^2) \qquad 0 \le x \le 1 \qquad \dots (5.10)$$

$$\hat{u}(t, 0) = \hat{u}(t, 1) = 0$$

Here $$\hat{f} = \frac{\partial^2 \hat{W}}{\partial t \, \partial x}$$

$$= \frac{1}{kh} \sum_{n=1}^{N} \sum_{j=1}^{J} \eta_{nj} \sqrt{kh}\, \chi_n(t)\, \chi_j(x)$$

where $\chi_n(t) = \begin{cases} 1, & t_{n-1} \le t \le t_n \\ 0, & \text{otherwise} \end{cases}$ $\chi_j(x) = \begin{cases} 1, & x_{j-1} \le x \le x_j \\ 0, & \text{otherwise} \end{cases}$

and $\eta_{nj} = \dfrac{1}{kh} \displaystyle\int_{t_{n-1}}^{t_n} \int_{x_{j-1}}^{x_j} dW(t, x) = N(0, 1)$

where $N(0, 1)$ is the standard real-valued Gaussian random variable and η_{nj} are independently and identically distributed.

Let $0 = t_0 < t_1 < \ldots < t_N = 1$ be the time partition of $[0, 1]$ and the time-step $k = \dfrac{1}{N}$.

Let $0 = x_0 < x_1 < x_2 < \ldots < x_j = 1$ be the space partition of $[0, 1]$ and the space-step $h = \dfrac{1}{J}$.

Let $U_j^n \approx \hat{u}(t_n, x_j)$ be the approximate solution of $\hat{u}(t_n, x_j)$.

We will approximate $\dfrac{\partial \hat{u}}{\partial t}$ at (t_n, x_j) by backward Euler method.

$$\frac{\partial \hat{u}(t_n, x_j)}{\partial t} \approx \frac{U_j^n - U_j^{n-1}}{k}$$

We will approximate $\dfrac{\partial \hat{u}(t_n, x_j)}{\partial x^2}$ at (t_n, x_j) by the central difference formula

$$\frac{\partial \hat{u}(t_n, x_j)}{\partial x^2} \approx \frac{U_{j-1}^n - 2U_j^n + U_{j+1}^n}{h^2}$$

Further, we have

$$g_j^n = g(t_n, x_j)$$

$$\hat{f}_j^n = \frac{1}{\sqrt{kh}} \eta_{jn}, \quad \eta_{jn} \sim N(0, 1)$$

The backward Euler method is defined by

$$\frac{U_j^n - U_j^{n-1}}{k} - \frac{U_{j-1}^n - 2U_j^n + U_{j+1}^n}{h^2} + bU_j^n = \hat{f}_j^n + g_j^n, \quad j = 1, 2, \ldots J.$$

$$U_j^0 = 10\, x_j^2\, (1 - x_j)^2, \quad j = 1, 2, \ldots, J - 1, \quad n = 1, 2, \ldots, N$$

Note that the Dirichlet boundary conditions implies that

$$U_0^n = U_J^n = 0, \quad n = 0, 1, ,2 \ldots, N$$

We can write the following matrix form with $\lambda = \frac{k}{h^2}$.

$$MU^n = KF^n + KG^n + U^{n-1}$$

Here

$$M = \begin{bmatrix} 1 + 2\lambda + kb & -\lambda & 0 & 0 & \cdots \\ -\lambda & 1 + 2\lambda + kb & -\lambda & 0 & \cdots \\ 0 & -\lambda & 1 + 2\lambda + kb & -\lambda & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & -\lambda \\ \cdots & \cdots & \cdots & -\lambda & 1 + 2\lambda + kb \end{bmatrix}_{(J-1)\times(J-1)}$$

$$F^n = \begin{bmatrix} \hat{f}_1^n \\ \hat{f}_2^n \\ \vdots \\ \hat{f}_{J-1}^n \end{bmatrix} \quad G^n = \begin{bmatrix} g_1^n \\ g_2^n \\ \vdots \\ g_{J-1}^n \end{bmatrix} \quad U^n = \begin{bmatrix} U_1^n \\ U_2^n \\ \vdots \\ U_{J-1}^n \end{bmatrix}$$

where U^n is the finite difference solution of $\hat{U}(t, x)$ at t_n with different x_j,

$j = 1, 2, \ldots, J - 1$.

By theorem we have for $t_n = 1$.

$$(E\,\|\,e^n\,\|^2)^{1/2} = \left(E\,\|\,U^n - u\,(t_n)\,\|_{L2}^2\right)^{1/2}$$

$$:= \left\{ E\left[\frac{1}{M} \sum_{j=1}^{M} \left|\, U_j^n - u\,(t_n, x_j)\, \right|^2 \right] \right\}^{1/2}$$

$$\therefore \qquad (E\,\|\,e^n\,\|^2)^{1/2} = \left\{ \frac{1}{N} \sum_{n=1}^{N} \left[\frac{1}{M} \sum_{j=1}^{M} \left|\, U_j^n\,(\omega_n) - u\,(t_n, x_j, \omega_n)\, \right|^2 \right] \right\}^{1/2}$$

$$= O\,(k^{1/4} + h^{1/2})$$

In the simulation we consider $N = 1000$ simulations. Next we will check the convergence order for k and h respectively as below.

Case 1 : Check the convergence order for k

Choose $h = 2^{-8}$

$$k_1 = 2^1,\quad k_2 = 2^2,\ \ldots,\quad k_{10} = 2^{10}$$

Then we get

$$(E\,\|\,e^n\,(k_1)\,\|^2)^{1/2},\ (E\,\|\,e^n\,(k_2)\,\|^2)^{1/2},\ \ldots,\ (E\,\|\,e^n\,(k_{10})\,\|^2)^{1/2}$$

Since h is very small, the error is dominated by k and we can neglect h.

The ratio of norms of the errors is

$$\frac{(E\,\|\,e^n\,(k_j)\,\|^2)^{1/2}}{(E\,\|\,e^n\,(k_{j+1})\,\|^2)^{1/2}} \approx \frac{k_j^{1/4}}{k_{j+1}^{1/4}} \approx \frac{\left(\frac{1}{2^j}\right)^{1/4}}{\left(\frac{1}{2^{j+1}}\right)^{1/4}} = 2^{1/4} \approx 1.1892$$

The table of results is as shown in Table-3.

Table-3

j	$\mathbf{k_j}$	**h**	$(E \parallel e^n (k_j) \parallel^2)^{1/2}$	$\dfrac{(E \parallel e^n (k_j) \parallel^2)^{1/2}}{(E \parallel e^n (k_{j+1}) \parallel^2)^{1/2}}$
1.000	0.5000	0.0039	0.3554	1.0154
2.0000	0.2500	0.0039	0.3500	1.0145
3.0000	0.1250	0.0039	0.3450	1.0299
4.0000	0.0625	0.0039	0.3350	1.0340
5.0000	0.0313	0.0039	0.3240	1.0351
6.0000	0.0156	0.0039	0.3130	1.0097
7.0000	0.0078	0.0039	0.3100	1.0333
8.0000	0.0039	0.0039	0.3000	1.0067
9.000	0.0020	0.0039	0.2980	1.0136
10.000	0.0010	0.0039	0.2940	1.0136

Since

$$(E \parallel e^n \parallel^2)^{1/2} \leq c\,(k^{1/4} + h^{1/2}) \approx ck^{1/4}$$

Taking logarithm of both sides of this inequality, we get

$$\log (E \parallel e^n \parallel^2)^{1/2} \leq \log c + \frac{1}{4} \log k$$

The graph of $\log (E \| e^n \|^2)^{1/2}$ against $\log k$ is a straight line having slope nearly equal to 0.2500 as shown in Fig. 5.5.

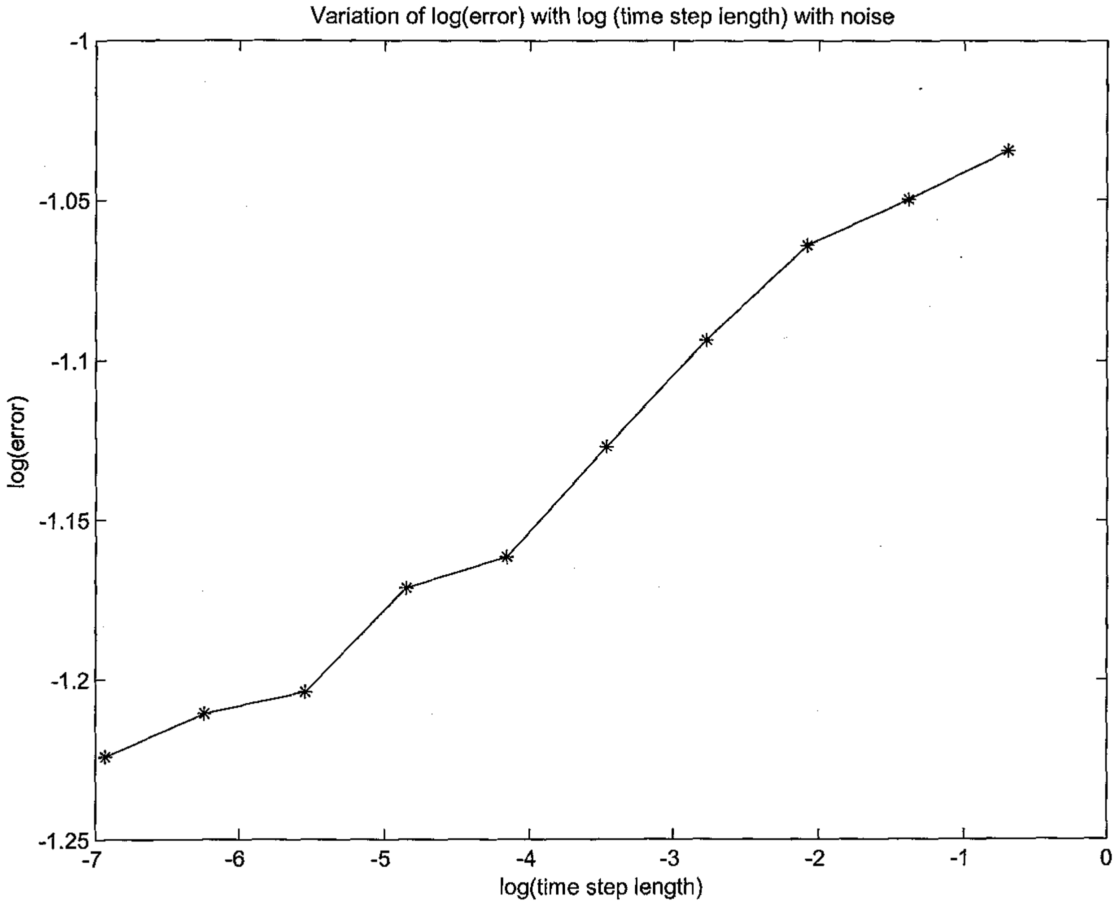

Fig. 5.5

Case 2 : Check the convergence order for h

$$\text{choose } k = 2^{-12}$$

$$h_1 = 2^{-2}, \; h_2 = 2^{-3}, \; \ldots, \; h_6 = 2^{-7}$$

The error is dominated by h as k is very small and we get

$$(E \| e^n (h_1) \|^2)^{1/2}, \; (E \| e^n (h_2) \|^2)^{1/2}, \; \ldots, \; (E \| e^n (h_6) \|^2)^{1/2}$$

The ratio of norms of the errors is

$$\frac{(E \,\|\, e^n (h_j) \,\|^2)^{1/2}}{(E \,\|\, e^n (h_{j+1}) \,\|^2)^{1/2}} \approx \frac{h_j^{1/2}}{h_{j+1}^{1/2}} \approx \frac{\left(\frac{1}{2^j}\right)^{1/2}}{\left(\frac{1}{2^{j+1}}\right)^{1/2}} = 2^{1/2} \approx 1.414$$

The corresponding results are as shown in below Table-4.

Table-4

j	**k**	$\mathbf{h_j}$	$\mathbf{(E \,\|\, e^n (h_j) \,\|^2)^{1/2}}$	$\mathbf{\dfrac{(E \,\|\, e^n (h_j) \,\|^2)^{1/2}}{(E \,\|\, e^n (h_{j+1}) \,\|^2)^{1/2}}}$
2.0000	0.0002	0.2500	0.4554	1.3578
3.0000	0.0002	0.1250	0.3354	1.5428
4.0000	0.0002	0.0625	0.2174	1.4209
5.0000	0.0002	0.0313	0.1530	1.2966
6.0000	0.0002	0.0156	0.1180	1.1683
7.0000	0.0002	0.0078	0.1010	1.1683

Since $(E \,\|\, e^n \,\|^2)^{1/2} \leq c\,(k^{1/4} + h^{1/2}) \approx ck^{1/2}$

Taking logarithm of both sides of this inequality, we get

$$\log (E \,\|\, e^n \,\|^2)^{1/2} \leq \log c + \frac{1}{2} \log h$$

The graph of $\log (E \| e^n \|^2)^{1/2}$ against $\log h$ is a straight line having slope nearly equal to 0.5000 as shown in Fig. 5.6.

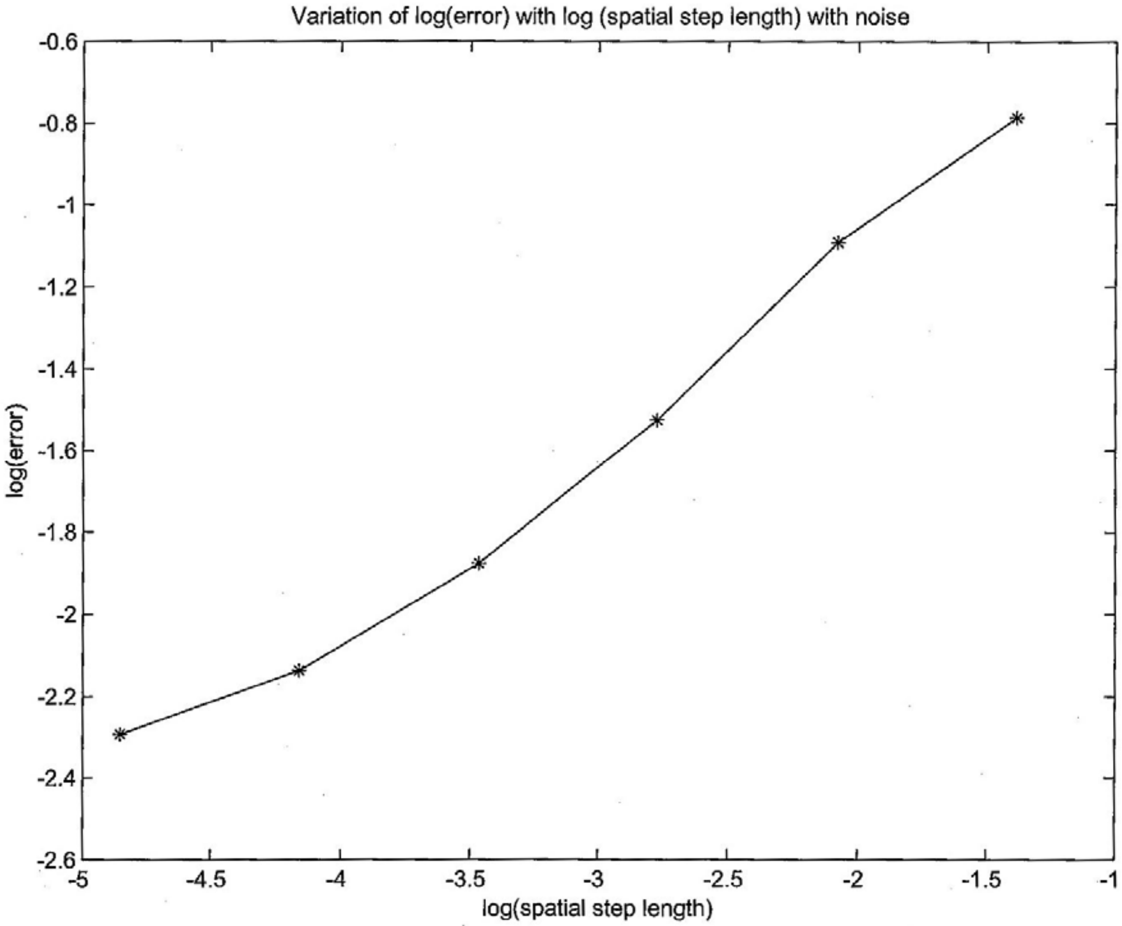

Fig. 5.6

CHAPTER 6

SUMMARY AND CONCLUSIONS

We discussed the finite difference method for the stochastic parabolic partial differential equations in this thesis. We first regularized the white noise by using the piece-wise constant random function. Then we obtained the error bounds in the L_2-norm between the exact solution and the approximate solution by using the **Green function** estimates. For the regularized parabolic partial differential equation we discretized the time derivative by using the forward Euler, backward Euler and Crank-Nicolson methods and we discretized the space derivative by using the central difference scheme. We obtained the error estimates in the L_2 norm.

Further we developed an algorithm to solve stochastic parabolic partial differential equation. We used this to solve test problems. We obtained numerical results for these problems and found that they are in accordance with theoretical results.

We only discussed 1D linear stochastic parabolic partial differential equation. The next step of this project is to extend the present approach to the 2D stochastic parabolic partial differential equations. In future, we will also consider the non-linear stochastic parabolic partial differential equations.

Another interesting future work of this project is to consider the error estimates for the stochastic parabolic partial differential equations with some special additive noise. In this case, we can expect better error bounds than these in case of white noise.

Finite difference method is restricted to rectangular domain. To consider the general domain in 2D, finite element method is more powerful and efficient. We will consider the finite element approximation for the stochastic parabolic partial differential equation in our future work.

APPENDIX

```
% Deterministic_h_variation

% To investigate the convergence order for the space step for the
stochastic parabolic problem.
% We choose
% space step k=1/(2^16)
% We choose different time steps
% h_j= 1/(2^j), j=2,3,4,5,6,7,8.
% The convergence order is O(h^2) for sufficiently small k.
clear

x=1;
T=1;
j_L2norm=[];
k_L2norm=[];
h_L2norm=[];
L2norm=[];

for jj=2:8    % different space steps
    jj
h=1/(2^jj);
n=x/h;
b0=0.5;

%Initial values U0
X0=(0:h:1)';
X0=X0(2:n);
U0=10*(X0.^2).*((1-X0).^2);

%The exact solution
exact=10*exp(1)* (X0.^2).*((1-X0).^2);

k=1/(2^16);
NT=T/k;
lambda=k/(h^2);
s=1;  %backward Euler method

% the approximation solution
a=1+2*s *lambda+k*b0;
b=-s*lambda;
%c=1-2*(1-s)*lambda;
%d=(1-s)*lambda;
aa=[];
 for i=1:n-1
aa=[aa a];
 end
bb=[];
  for i=1:n-2
bb=[bb b];
  end
```

```
% Deterministic_k_variation
% To investigate the convergence order for the time step for the
stochastic parabolic problem.
% We choose
% space step h=1/(2^10)
% We choose different time steps
% k_j= 1/(2^j), j=1,2,3,4,5,6,7,8
% The convergence order is O(k) for sufficiently small h.

clear
x=1;
T=1;
h=1/(2^10);
n=x/h;
b0=0.5;

j_L2norm=[];
k_L2norm=[];
h_L2norm=[];
L2norm=[];

%Initial values U0
X0=(0:h:1)';
X0=X0(2:n);
U0=10*(X0.^2).*((1-X0).^2);

%The exact solution
exact=10*exp(1)* (X0.^2).*((1-X0).^2);


for jj=1:8   % different time steps

k=1/(2^jj);
NT=T/k;
lambda=k/(h^2);
s=1;  %backward Euler method
% the approximation solution
a=1+2*s *lambda+k*b0;
b=-s*lambda;
%c=1-2*(1-s)*lambda;
%d=(1-s)*lambda;
aa=[];
 for i=1:n-1
aa=[aa a];
 end
bb=[];
  for i=1:n-2
bb=[bb b];
  end
```

```
M=diag(aa)+diag(bb,1)+diag(bb,-1);
%cc=[];
%for i=1:J-1;
%cc=[cc c];
%end
%dd=[];
%for i=1:J-2
%dd=[dd d];
%end
%B=diag(cc)+diag(dd,1)+diag(dd,-1);

% the approximation of the Wiener process W(t,x)
 for j=1:NT          %consider one time step
        F_n = 1/sqrt(k*h) * randn(n-1,1);
    G_n= 10 *(1+b0)*(X0.^2).*((1-X0).^2)*exp(j*k) -10*(2-12*X0
+12*X0.^2)*exp(j*k);
    U1=M\(0*k*F_n + k*G_n +U0);
    U0=U1;
    end

     j_L2norm=[j_L2norm; jj];
     k_L2norm=[k_L2norm;k];
     h_L2norm=[h_L2norm;h];
     L2norm=[L2norm;norm(U0-exact)*(h^(1/2))];

end
ratio_k_L2norm=[];
    for c=1:length(L2norm)-1
ratio_k_L2norm=[ratio_k_L2norm; L2norm(c)/L2norm(c+1)];
    end
ratio_k_L2norm=[ratio_k_L2norm; ratio_k_L2norm(length(L2norm)-1)];

%figure
figure(1)
plot(log(k_L2norm),log(L2norm))
hold on
plot(log(k_L2norm),log(L2norm), '*')

title('Variation of log(error) with log (time step length) without
noise')
xlabel('log(time step length)')
ylabel('log(error)')
xx=log(k_L2norm); yy= log(L2norm);
tangent_of_the_line= (yy(3)-yy(2))/(xx(3)-xx(2));

%table
[j_L2norm k_L2norm h_L2norm L2norm ratio_k_L2norm]
```

```
% Stochastic_h_variation
% To investigate the convergence order for the space step for the
stochastic parabolic problem.
% We choose
% space step k=1/(2^12)
% We choose different time steps
% h_j= 1/(2^j), j=2,3,4,5,6,7.
% The convergence order is O(h^(1/2)) for sufficiently small k.

clear
x=1;
T=1;
b0=0.5;

N=1000;     %N is the number of the different realization
N_L2norm=[];

for l=1:N
%The exact solution
 k=1/(2^12);
 h0=1/(2^7);
n=x/h0;

%Initial values U0
X0=(0:h0:1)';
X0=X0(2:n);
U0=10*(X0.^2).*((1-X0).^2);

NT=T/k;

lambda=k/(h0^2);
s=1;  %backward Euler method

% the approximation solution
a=1+2*s *lambda+k*b0;
b=-s*lambda;
%c=1-2*(1-s)*lambda;
%d=(1-s)*lambda;
aa=[];
 for i=1:n-1
aa=[aa a];
 end
bb=[];
  for i=1:n-2
bb=[bb b];
  end
M=diag(aa)+diag(bb,1)+diag(bb,-1);

    for j=1:NT          %consider one time step
        F_n = 1/sqrt(k*h0) * randn(n-1,1);    %stochastic term
    G_n= 10 *(1+b0)*(X0.^2).*((1-X0).^2)*exp(j*k) -10*(2-12*X0
+12*X0.^2)*exp(j*k);
    U1=M\(k*F_n + k*G_n +U0);
    U0=U1;
    end
```

```
exact=U0;
jjj=7;
L2norm=[];

for jj=2:jjj   % different time steps
h=1/(2^jj);

n=x/h;

%Initial values U0
X0=(0:h:1)';
X0=X0(2:n);
U0=10*(X0.^2).*((1-X0).^2);

NT=T/k;

lambda=k/(h^2);
s=1;  %backward Euler method

% the approximation solution
a=1+2*s *lambda+k*b0;
b=-s*lambda;

aa=[];
 for i=1:n-1
aa=[aa a];
 end
bb=[];
  for i=1:n-2
bb=[bb b];
  end
M=diag(aa)+diag(bb,1)+diag(bb,-1);

for j=1:NT          %consider one time step
        F_n = 1/sqrt(k*h) * randn(n-1,1);    %stochastic term
    G_n= 10 *(1+b0)*(X0.^2).*((1-X0).^2)*exp(j*k) -10*(2-12*X0
+12*X0.^2)*exp(j*k);
    U1=M\(k*F_n + k*G_n +U0);
    U0=U1;
    end

    exact0=exact((h/h0)*(1:2^jj-1));
    L2norm0=norm(U0-exact0)*(h^(1/2));
    L2norm=[L2norm;L2norm0];

end
N_L2norm=[N_L2norm L2norm];

end
 j_L2norm=[];
k_L2norm=[];
h_L2norm=[];
Expec_L2norm=[];
```

```
for jj=1:jjj-1
     j_L2norm=[j_L2norm; jj+1];
     k_L2norm=[k_L2norm;k];
     h_L2norm=[h_L2norm;1/(2^(jj+1))];
     Expec_L2norm=[Expec_L2norm; ((1/N)*sum(N_L2norm(jj,:).^2))^(1/2)];
end

ratio_k_Expec_L2norm=[];
    for c=1:length(Expec_L2norm)-1
ratio_k_Expec_L2norm=[ratio_k_Expec_L2norm;
Expec_L2norm(c)/Expec_L2norm(c+1)];
    end
ratio_k_Expec_L2norm=[ratio_k_Expec_L2norm;
ratio_k_Expec_L2norm(length(Expec_L2norm)-1)];

%figure
figure(1)
title('Variation of log(error) with log (spatial step length) with
noise')
plot(log(h_L2norm),log(Expec_L2norm))
hold on
plot(log(h_L2norm),log(Expec_L2norm),'*')
title('Variation of log(error) with log (spatial step length) with
noise')
xlabel('log(spatial step length)')
ylabel('log(error)')
%table
[j_L2norm k_L2norm h_L2norm Expec_L2norm ratio_k_Expec_L2norm]
```

A.4 Stochastic_k_Variation

```
% Stochastic_k_variation

% To investigate the convergence order for the time step for the
stochastic parabolic problem.
% We choose
% space step h=1/(2^8)
% We choose different time steps
% k_j= 1/(2^j), j=1,2,3,4,5,6,7,8,9,10
% The convergence order is O(k^(1/4)) for sufficiently small h.
clear
x=1;
T=1;
h=1/(2^8);
n=x/h;
b0=0.5;
%Initial values U0
X0=(0:h:1)';
X0=X0(2:n);
U0=10*(X0.^2).*((1-X0).^2);
U00=U0;

N=1000;     %N is the number of the different realization
N_L2norm=[];

for l=1:N
    l
    U0=U00;
%The exact solution
 k=1/(2^8);

NT=T/k;
lambda=k/(h^2);
s=1;  %backward Euler method

% the approximation solution
a=1+2*s *lambda+k*b0;
b=-s*lambda;
%c=1-2*(1-s)*lambda;
%d=(1-s)*lambda;
aa=[];
 for i=1:n-1
aa=[aa a];
 end
bb=[];
  for i=1:n-2
bb=[bb b];
  end
M=diag(aa)+diag(bb,1)+diag(bb,-1);

    for j=1:NT          %consider one time step
        F_n = 1/sqrt(k*h) * randn(n-1,1);    %stochastic term
    G_n= 10 *(1+b0)*(X0.^2).*((1-X0).^2)*exp(j*k) -10*(2-12*X0
+12*X0.^2)*exp(j*k);
    U1=M\(k*F_n + k*G_n +U0);
    U0=U1;
    end
```

```
exact=U0;
jjj=10;
L2norm=[];

for jj=1:jjj   % different time steps
k=1/(2^jj);

NT=T/k;
lambda=k/(h^2);
s=1;  %backward Euler method

% the approximation solution
a=1+2*s *lambda+k*b0;
b=-s*lambda;

aa=[];
 for i=1:n-1
aa=[aa a];
 end
bb=[];
  for i=1:n-2
bb=[bb b];
  end
M=diag(aa)+diag(bb,1)+diag(bb,-1);
for j=1:NT    %consider one time step
    F_n = 1/sqrt(k*h) * randn(n-1,1);    %stochastic term
    G_n= 10 *(1+b0)*(X0.^2).*((1-X0).^2)*exp(j*k) -10*(2-12*X0
+12*X0.^2)*exp(j*k);
    U1=M\(k*F_n + k*G_n +U0);
    U0=U1;
    end

    L2norm0=norm(U0-exact)*(h^(1/2));
    L2norm=[L2norm;L2norm0];
end
N_L2norm=[N_L2norm L2norm];
end

j_L2norm=[];
k_L2norm=[];
h_L2norm=[];
Expec_L2norm=[];

for jj=1:jjj
     j_L2norm=[j_L2norm; jj];
     k_L2norm=[k_L2norm;1/(2^jj)];
     h_L2norm=[h_L2norm;h];
     Expec_L2norm=[Expec_L2norm; ((1/N)*sum(N_L2norm(jj,:).^2))^(1/2)];
end
```

```
ratio_k_Expec_L2norm=[];
    for c=1:length(Expec_L2norm)-1
ratio_k_Expec_L2norm=[ratio_k_Expec_L2norm;
Expec_L2norm(c)/Expec_L2norm(c+1)];
    end
ratio_k_Expec_L2norm=[ratio_k_Expec_L2norm;
ratio_k_Expec_L2norm(length(Expec_L2norm)-1)];

%figure
figure(1)
plot(log(k_L2norm),log(Expec_L2norm))
hold on
plot(log(k_L2norm),log(Expec_L2norm),'*')

title('Variation of log(error) with log (time step length) with noise')
xlabel('log(time step length)')
ylabel('log(error)')
%table
[j_L2norm k_L2norm h_L2norm Expec_L2norm ratio_k_Expec_L2norm]
```